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Sigma-Model Solitons in the Noncommutative Plane: Construction and Stability Analysis

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Abstract

Noncommutative multi-solitons are investigated in Euclidean two-dimensional U(n) and Grassmannian sigma models, using the auxiliary Fock-space formalism. Their construction and moduli spaces are reviewed in some detail, unifying abelian and nonabelian configurations. The analysis of linear perturbations around these backgrounds reveals an unstable mode for the U(n) models but shows stability for the Grassmannian case. For multi-solitons which are diagonal in the Fock-space basis we explicitly evaluate the spectrum of the Hessian and identify all zero modes. It is very suggestive but remains to be proven that our results qualitatively extend to the entire multi-soliton moduli space.

Contents

1 Introduction

Multi-solitons in noncommutative Euclidean two-dimensional sigma models are of interest as static D0-branes inside D2-branes with a constant B-field background [1] but also per se as nonperturbative classical field configurations. Adding a temporal dimension, they represent static solutions not only of a (WZW-extended) sigma model but also of the Yang-Mills-Higgs BPS equations on noncommutative $\mathbb{R}^{2+1}$. In fact, the full BPS sector of the Yang-Mills-Higgs system is, in a particular gauge, given by the (time-dependent) solutions to the sigma-model equations of motion.

In the commutative case, the classical solutions of Euclidean two-dimensional sigma models have been investigated intensively by physicists as well as by mathematicians (for a review see, e.g. [2]). Prominent target spaces are $\mathrm{U}(n)$ group manifolds or their Grassmannian cosets $\mathrm{Gr}(n,r) = \frac{\mathrm{U}(n)}{\mathrm{U}(r)\times\mathrm{U}(n-r)}$ for $1 \le r < n$ (which are geodesic submanifolds of $\mathrm{U}(n)$).[1] Any classical solution Φ for the $\mathrm{U}(n)$ sigma model can be constructed iteratively, with at most $n-1$ so-called unitons as building blocks [3, 4]. The subset of hermitian solutions, $\Phi^\dagger = \Phi$, coincides with the space of Grassmannian solutions, $\Phi \in \mathrm{Gr}(n,r)$ for some rank $r < n$, of which again a subset is distinguished by a BPS property.[2] These BPS configurations are precisely the one-uniton solutions and can be interpreted as (static) multi-solitons.

Let us now perturb such multi-solitons within the configuration space of the two-dimensional (i.e. static) sigma-model, either within their Grassmannian or, more widely, within the whole group manifold. A linear stability analysis then admits a two-fold interpretation. First, it is relevant for the semiclassical evaluation of the Euclidean path integral, revealing potential *quantum* instabilities of the two-dimensional model. Second, it yields the (infinitesimal) time evolution of fluctuations around the static multi-soliton in the time-extended three-dimensional theory, indicating *classical* instabilities if they are present. More concretely, any static perturbation of a classical configuration can be taken as (part of the) Cauchy data for a classical time evolution, and any negative eigenvalue of the quadratic fluctuation operator will give rise to an exponential runaway behavior, at least within the linear response regime. Furthermore, fluctuation zero modes are expected to belong to moduli perturbations of the classical configuration under consideration. The current knowledge on the effect of quantum fluctuations is summarized in [2].

The Moyal deformation of Euclidean two-dimensional sigma models has been described in [5], including the construction of their BPS solutions. The operator formulation of the noncommutative $\mathrm{U}(n)$ theory turns it into a (zero-dimensional) matrix model with $\mathrm{U}(\mathbb{C}^n \otimes \mathcal{H})$ as its target space, where $\mathcal{H}$ denotes a Heisenberg algebra representation module. Each BPS configuration again belongs to some Grassmannian, whose rank r can be finite or infinite. The latter case represents a smooth deformation of the known commutative multi-solitons, while the former situation realizes a noncommutative novelty, namely abelian multi-solitons (they even occur for the $\mathrm{U}(1)$ model). Although these solutions are available in explicit form, very little is known about their stability.[3] Our work sheds some light on this issue, in particular for the interesting abelian case.

The paper is organized as follows. In the next section we review the noncommutative two-dimensional $\mathrm{U}(n)$ sigma-model and present nested classes of finite-energy solutions to its equation of motion. We focus on BPS configurations (multi-solitons) and their moduli spaces, unifying the previously different descriptions of abelian and nonabelian multi-solitons. Section 3 is devoted to a study of the fluctuations around generic multi-solitons and detects a universal unstable mode.

[1] Most familiar are the $\mathrm{Gr}(n,1) = \mathbb{C}P^{n-1}$ models.

[2] not to be confused with the BPS condition for the Yang-Mills-Higgs system

[3] A few recent works [6, 7, 8, 9, 10, 11] address perturbations of noncommutative solitons but not for sigma models.

Section 4 analyzes the spectrum of the Hessian specifically for U(1) backgrounds and settles the issue for diagonal BPS solutions, analytically as well as numerically. Section 5 addresses the fluctuation problem in the nonabelian case for the example of the noncommutative U(2) model. We close with a summary and a list of open questions.

2 Noncommutative 2d sigma model and its solutions

Before addressing fluctuations, it is necessary to present the noncommutative two-dimensional sigma model and its multi-soliton solutions which we will set out to perturb later on. These solutions have been discussed as static solutions of a 2+1 dimensional noncommutative sigma model [5, 12, 13, 14] which results from Moyal deforming the WZW-modified integrable sigma model [15, 16, 17]. Here we review the results pertaining to the static situation.

2.1 Noncommutativity

A Moyal deformation of Euclidean $\mathbb{R}^2$ with coordinates (x, y) is achieved by replacing the ordinary pointwise product of smooth functions on it with the noncommutative but associative Moyal star product. The latter is characterized by a constant positive real parameter θ which prominently appears in the star commutation relation between the coordinates,

$$x \star y - y \star x \;\equiv\; [\,x\,,\,y\,]_\star \;=\; \mathrm{i}\,\theta\;. \tag{2.1}$$

For a concise treatment of the Moyal star product see [18, 19, 20]. It is convenient to work with the complex combinations

$$z \;=\; x+\mathrm{i}y \quad\text{and}\quad \bar{z} \;=\; x-\mathrm{i}y \qquad\Longrightarrow\qquad [\,z\,,\,\bar{z}\,]_\star \;=\; 2\theta \tag{2.2}$$

and to scale them to

$$a \;=\; \tfrac{z}{\sqrt{2\theta}} \quad\text{and}\quad a^\dagger \;=\; \tfrac{\bar{z}}{\sqrt{2\theta}} \qquad\Longrightarrow\qquad [\,a\,,\,a^\dagger\,]_\star \;=\; 1\;. \tag{2.3}$$

A different realization of this Heisenberg algebra promotes the coordinates (and thus all their functions) to noncommuting operators acting on an auxiliary Fock space $\mathcal{H}$ but keeps the ordinary operator product. The Fock space is a Hilbert space with orthonormal basis states

$$\begin{gathered} |m\rangle \;=\; \tfrac{1}{\sqrt{m!}}\,(a^\dagger)^m\,|0\rangle \qquad\text{for}\quad m\in\mathbb{N}_0 \quad\text{and}\quad a|0\rangle=0\;, \\ a\,|m\rangle \;=\; \sqrt{m}\,|m{-}1\rangle\;,\qquad a^\dagger\,|m\rangle \;=\; \sqrt{m{+}1}\,|m{+}1\rangle\;,\qquad N\,|m\rangle \;:=a^\dagger a\,|m\rangle \;=\; m\,|m\rangle\;, \end{gathered} \tag{2.4}$$

therewith characterizing a and $a^\dagger$ as standard annihilation and creation operators. The star-product and operator formulations are tightly connected through the Moyal-Weyl map: Coordinate derivatives correspond to commutators with coordinate operators,

$$\sqrt{2\theta}\,\partial_z \;\leftrightarrow\; -\mathrm{ad}(a^\dagger) \quad,\qquad \sqrt{2\theta}\,\partial_{\bar{z}} \;\leftrightarrow\; \mathrm{ad}(a) \quad, \tag{2.5}$$

and the integral over the noncommutative plane reads

$$\int \mathrm{d}^2x\; f_\star(x) \;=\; 2\pi\,\theta\,\mathrm{Tr}_{\mathcal{H}}\,f_{\mathrm{op}}\;, \tag{2.6}$$

where the function $f_\star$ corresponds to the operator f_{op} via the Moyal-Weyl map and the trace is over the Fock space $\mathcal{H}$. We shall work with the operator formalism but refrain from introducing special notation indicating operators, so all objects are operator-valued if not said otherwise.

2.2 Two-dimensional sigma model

The fields Φ of the noncommutative two-dimensional U(n) sigma model are unitary $n{\times}n$ matrices with operator-valued entries, i.e. $\Phi \in \mathrm{U}(\mathbb{C}^n \otimes \mathcal{H}) = \mathrm{U}(\mathcal{H}^{\oplus n})$, and thus subject to the constraint

$$\Phi\,\Phi^\dagger \;=\; \mathbb{1}_n \otimes \mathbb{1}_{\mathcal{H}} \;=\; \Phi^\dagger\,\Phi\ . \tag{2.7}$$

The Euclidean action of the model coincides with the energy functional of its 2+1 dimensional extension evaluated on static configurations [5],[4]

$$\begin{aligned} E[\Phi] &= 2\pi\,\theta\,\mathrm{Tr}\big(\partial_z\Phi^\dagger\,\partial_{\bar z}\Phi + \partial_{\bar z}\Phi^\dagger\,\partial_z\Phi\big) \\ &= \pi\,\mathrm{Tr}\big([a,\Phi]^\dagger\,[a,\Phi] + [a,\Phi^\dagger][a,\Phi^\dagger]^\dagger\big) \\ &= 2\,\pi\,\mathrm{Tr}\big([a,\Phi]^\dagger\,[a,\Phi]\big) \;=\; 2\,\pi\,\big|[a,\Phi]\big|^2 \\ &= \pi\,\mathrm{Tr}\big(\Delta\Phi^\dagger\,\Phi + \Phi^\dagger\Delta\Phi\big)\ , \end{aligned} \tag{2.8}$$

where the trace is taken over the Fock space $\mathcal{H}$ as well as over the U(n) group space. Here, we have introduced the hermitian Laplace operator $-\Delta$ which is defined via

$$\Delta\mathcal{O} \;:=\; \big[a,[a^\dagger,\mathcal{O}]\big] \;=\; \big[a^\dagger,[a,\mathcal{O}]\big]\ . \tag{2.9}$$

Its kernel is spanned by functions only of a or only of $a^\dagger$. Varying $E[\Phi]$ under the above constraint one finds the equation of motion,[5]

$$0 \;=\; \big[a\,,\Phi^\dagger[a^\dagger,\Phi]\big] + \big[a^\dagger,\Phi^\dagger[a\,,\Phi]\big] \;=\; \Phi^\dagger\,\Delta\Phi - \Delta\Phi^\dagger\,\Phi\ . \tag{2.10}$$

Since the annihilation and creation operators above should be read as $\mathbb{1}_n \otimes a$ and $\mathbb{1}_n \otimes a^\dagger$, respectively, this model enjoys a global U(1)$\times$SU(n)$\times$SU(n) invariance under

$$\Phi \quad\longrightarrow\quad (V \otimes \mathbb{1}_{\mathcal{H}})\,\Phi\,(W \otimes \mathbb{1}_{\mathcal{H}}) \tag{2.11}$$

with $VV^\dagger = \mathbb{1}_n = WW^\dagger$, which generates a moduli space to any nontrivial solution of (2.10). Another obvious symmetry is induced by the ISO(2) Euclidean group transformations of the noncommutative plane, as generated by the adjoint action of a, $a^\dagger$ and N. Specifically, a global translation of the noncommutative plane,

$$(z\,,\,\bar z) \quad\longrightarrow\quad (z+\zeta\,,\,\bar z+\bar\zeta) \;=\; \sqrt{2\theta}\,(a+\alpha\,,\,a^\dagger+\bar\alpha)\ , \tag{2.12}$$

induces on (operator-valued) scalar functions f the unitary operation

$$f \quad\longrightarrow\quad \mathrm{e}^{-\zeta\partial_z-\bar\zeta\partial_{\bar z}}\,f \;=\; \mathrm{e}^{\alpha\,\mathrm{ad}(a^\dagger)-\bar\alpha\,\mathrm{ad}(a)}\,f \;=\; \mathrm{e}^{\alpha a^\dagger-\bar\alpha a}\,f\,\mathrm{e}^{-\alpha a^\dagger+\bar\alpha a} \;=:\; D(\alpha)\,f\,D(\alpha)^\dagger\ . \tag{2.13}$$

Its action on states leads to coherent states, e.g.

$$D(\alpha)\,|0\rangle \;=\; \mathrm{e}^{\alpha a^\dagger-\bar\alpha a}\,|0\rangle \;=\; \mathrm{e}^{-\frac12\bar\alpha\alpha}\,\mathrm{e}^{\alpha a^\dagger}\,|0\rangle \;=:\; |\alpha\rangle\ . \tag{2.14}$$

A global rotation of the noncommutative plane,

$$(z\,,\,\bar z) \quad\longrightarrow\quad (\mathrm{e}^{\mathrm{i}\,\vartheta}z\,,\,\mathrm{e}^{-\mathrm{i}\,\vartheta}\bar z) \;=\; \sqrt{2\theta}\,(\mathrm{e}^{\mathrm{i}\,\vartheta}a\,,\,\mathrm{e}^{-\mathrm{i}\,\vartheta}a^\dagger)\ , \tag{2.15}$$

[4] $|A|^2 = \mathrm{Tr}(A^\dagger A)$ is the squared Hilbert-Schmidt norm of U(n)-valued operators on $\mathcal{H}$. We restrict ourselves to finite-energy configurations, i.e. we demand that $[a,\Phi]$ exists and is Hilbert-Schmidt. Furthermore we only consider solutions for which $\Delta\Phi$ is traceclass and Φ is a bounded operator in order for the below expressions to be well defined.

[5] assuming that the appearing total derivative terms vanish

induces the unitary transformation

$$f \quad \longrightarrow \quad \mathrm{e}^{\mathrm{i}\vartheta\,\mathrm{ad}(a^\dagger a)}\, f \;=\; \mathrm{e}^{\mathrm{i}\vartheta\, a^\dagger a}\, f\, \mathrm{e}^{-\mathrm{i}\vartheta\, a^\dagger a} \;=:\; R(\vartheta)\, f\, R(\vartheta)^\dagger \tag{2.16}$$

for $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, because $[N, f] = \bar{z}\partial_{\bar{z}} f - \partial_z f z$. Applying $R(\vartheta)$ to a coherent state we obtain

$$R(\vartheta)|\alpha\rangle \;=\; |\mathrm{e}^{\mathrm{i}\,\vartheta}\alpha\rangle \;. \tag{2.17}$$

Since the adjoint actions of both $D(\alpha)$ and $R(\vartheta)$ commute with that of Δ, the energy functional is invariant under them, and all translates and rotations of a solution to (2.10) also qualify as solutions, with equal energy. However, other unitary transformations will in general change the value of E.

There is a wealth of finite-energy configurations Φ which fulfil the equation of motion (2.10). A subclass of those is distinguished by possessing a smooth commutative limit, in which Φ merges with a commutative solution. The latter have been classified by [3, 4]. We call these "nonabelian" because they cannot appear in the U(1) case, where the commutative model is a free field theory and does not admit finite-energy solutions. Yet, there can (and do) exist non-trivial abelian finite-energy solutions at finite θ, whose $\theta \to 0$ limit is necessarily singular and produces a discontinuous configuration. We term such solutions "abelian" even in case they are imbedded in a nonabelian group.

Of particular interest are configurations diagonal in the oscillator basis (2.4) as well as in $\mathbb{C}^n$, namely $\Phi = \bigoplus_{i=1}^n \mathrm{diag}\big(\{\mathrm{e}^{\mathrm{i}\alpha_i^\ell}\}_{\ell=0}^\infty\big)$ with $\alpha_i^\ell \in \mathbb{R}$. A short computation reveals that such a configuration obeys the equation of motion (2.10) if and only if the phases satisfy $\mathrm{e}^{\mathrm{i}\alpha_i^\ell} = \pm \mathrm{e}^{\mathrm{i}\alpha_i}$ where the sign depends on ℓ. Its energy is finite if the number of positive signs or the number of negative signs in each block labelled by i is finite.

2.3 Grassmannian configurations

It is a daunting task to classify all solutions to the full equation of motion (2.10), and we shall focus on the subset of hermitian ones. Any (not necessarily classical) hermitian configuration obeys

$$\Phi^\dagger \;=\; \Phi \qquad \Longrightarrow \qquad \Phi^2 \;=\; \mathbb{1}_n \otimes \mathbb{1}_{\mathcal{H}} \;=:\; \mathbb{1} \;, \tag{2.18}$$

and is conveniently parametrized by a hermitian projector $P{=}P^\dagger$ via

$$\Phi \;=:\; \mathbb{1} - 2P \;=\; P^\perp - P \;=\; \mathrm{e}^{\mathrm{i}\pi P} \qquad \text{with} \quad P^2 = P \;, \tag{2.19}$$

where $P^\perp = \mathbb{1}{-}P$ denotes the complementary projector. This allows one to define a topological charge like in the commutative case as [21]

$$\begin{aligned} Q[\Phi] \;&=\; \tfrac14\,\theta\,\mathrm{Tr}\big(\Phi\,\partial_z\Phi\,\partial_{\bar z}\Phi - \Phi\,\partial_{\bar z}\Phi\,\partial_z\Phi\big) \\ &=\; \mathrm{Tr}\big(P\,[a^\dagger, P]\,[a\,, P] - P\,[a\,, P]\,[a^\dagger, P]\big) \\ &=\; \big|P\,[a\,, P]\big|^2 - \big|[a\,, P]\,P\big|^2 \\ &=\; \mathrm{Tr}\big(P - P\,a\,P\,a^\dagger P + P\,a^\dagger P\,a\,P\big) \\ &=\; \mathrm{Tr}\big(P\,a\,(\mathbb{1}{-}P)\,a^\dagger P - P\,a^\dagger(\mathbb{1}{-}P)\,a\,P\big) \;, \end{aligned} \tag{2.20}$$

which may be compared with the energy

$$
\begin{aligned}
\tfrac{1}{8\pi}\,E[\Phi] &= \tfrac12\,\theta\,\mathrm{Tr}\big(\partial_z\Phi\,\partial_{\bar z}\Phi\big) \\
&= \mathrm{Tr}\big([a^\dagger,P]\,[P\,,a]\big) \\
&= \big|P\,[a\,,P]\big|^2+\big|[a\,,P]\,P\big|^2 \\
&= \mathrm{Tr}\big(P\,(a\,a^\dagger{+}a^\dagger a)\,P-P\,a\,P\,a^\dagger P-P\,a^\dagger P\,a\,P\big) \\
&= \mathrm{Tr}\big(P\,a\,(\mathbb{1}{-}P)\,a^\dagger P+P\,a^\dagger(\mathbb{1}{-}P)\,a\,P\big)\ .
\end{aligned}
\tag{2.21}
$$

Each hermitian projector P gives rise to the complementary projector $\mathbb{1}{-}P$, with the properties[6]

$$
E[\mathbb{1}{-}P] \;=\; E[P] \qquad\text{and}\qquad Q[\mathbb{1}{-}P] \;=\; -Q[P]\ . \tag{2.22}
$$

Comparing (2.20) and (2.21) we get the relations

$$
\tfrac{1}{8\pi}\,E[P] \;=\; Q[P]+2\big|[a\,,P]\,P\big|^2 \;=\; -Q[P]+2\big|P\,[a\,,P]\big|^2\ , \tag{2.23}
$$

which yield the BPS bound

$$
E[P] \;\ge\; 8\pi\,\big|Q[P]\big| \tag{2.24}
$$

for any hermitian configuration.

The set of all projectors unitarily equivalent to P is called the Grassmannian, and it is given by the coset

$$
\mathrm{Gr}(P) \;=\; \frac{\mathrm{U}(\mathcal{H}^{\oplus n})}{\mathrm{U}(\mathrm{im}P)\times\mathrm{U}(\mathrm{ker}P)}\ . \tag{2.25}
$$

Thus, each given P (and thus hermitian Φ) belongs to a certain Grassmannian, and the space of all hermitian Φ decomposes into a disjoint union of Grassmannians. The restriction to hermitian Φ reduces the unitary to a Grassmannian sigma model, whose configuration space is parametrized by projectors P. In the commutative case the topological charge Q is an element of $\pi_2(\mathrm{Gr}(P))=\mathbb{Z}$, where S^2 is the compactified plane. Hence, it is an invariant of the Grassmannian if one excludes from (2.25) any singular unitary transformations with nontrivial winding at infinity. It is less obvious how to properly extend this consideration to the infinite-dimensional cases encountered here [22]. We therefore take a pragmatic viewpoint and demand $Q[P]$ to be constant throughout the Grassmannian. This may downsize the above coset $\mathrm{Gr}(P)$ by restricting the set of admissible unitaries U.

Let us make this more explicit by computing $Q[UPU^\dagger]$. To this end, we define

$$
\omega \;:=\; U^\dagger\,[a\,,U] \qquad\text{and}\qquad \omega^\dagger \;=\; U^\dagger\,[a^\dagger,U]\ , \qquad\text{with}\qquad [a+\omega\,,a^\dagger+\omega^\dagger] \;=\; \mathbb{1}\ , \tag{2.26}
$$

as elements of the Lie algebra of $\mathrm{U}(\mathcal{H}^{\oplus n})$.[7] A short calculation yields

$$
\begin{aligned}
Q[UPU^\dagger] &= \mathrm{Tr}\big(P\,[a^\dagger{+}\omega^\dagger,P]\,[a{+}\omega\,,P]-P\,[a{+}\omega\,,P]\,[a^\dagger{+}\omega^\dagger,P]\big) \qquad (2.27)\\
&= Q[P]\;+\;\mathrm{Tr}\,P\,\big([a^\dagger,P]\omega+\omega[a^\dagger,P]-[a,P]\omega^\dagger-\omega^\dagger[a,P]-\omega^\dagger(\mathbb{1}{-}P)\omega+\omega(\mathbb{1}{-}P)\omega^\dagger\big)P\ ,
\end{aligned}
$$

[6] By a slight abuse of notation, we denote the energy and topological charge as functionals of P again with the symbols E and Q, respectively.

[7] In fact, the finite-energy condition (see footnote before (2.8)) enforces $[a\,,UPU^\dagger]$ to be Hilbert-Schmidt which implies that $[\omega\,,P]$ is Hilbert-Schmidt as well. Unfortunately, this does not suffice to guarantee the constancy of Q in $\mathrm{Gr}(P)$. On the other hand, ω itself need not even be bounded, as in the example of (2.16) where $\omega=(\mathrm{e}^{\mathrm{i}\vartheta}{-}1)a$.

which constrains ω in terms of P. In case of $\mathrm{Tr}P < \infty$, this indeed reduces to

$$Q[UPU^\dagger] \;=\; Q[P] \;+\; \mathrm{Tr}\,P\left([\omega\,,a^\dagger]+[a\,,\omega^\dagger]+[\omega\,,\omega^\dagger]\right) \;=\; Q[P]\;. \tag{2.28}$$

Alternatively, we may calculate the infinitesimal variation of $Q[P]$ under $P\to P{+}\delta P$. Remembering that P and δP are bounded and that their commutators with a or $a^\dagger$ are Hilbert-Schmidt, we get

$$\delta Q[P] \;=\; \mathrm{Tr}\big(\big[a^\dagger,[P,\delta P][a,P]\big]-\big[a,[P,\delta P][a^\dagger,P]\big]\big) \;+\; 3\,\mathrm{Tr}\,\delta P\big([a^\dagger,P][a,P]-[a,P][a^\dagger,P]\big) \tag{2.29}$$

with the first trace being a "boundary term". Variations inside the Grassmannian are given by

$$\delta P \;=\; [\Lambda_\mathrm{o}\,,P] \qquad\text{with}\qquad \Lambda_\mathrm{o}^\dagger=-\Lambda_\mathrm{o} \qquad\text{and}\qquad P\Lambda_\mathrm{o}P \;=\; 0 \;=\; (\mathbb{1}{-}P)\Lambda_\mathrm{o}(\mathbb{1}{-}P) \tag{2.30}$$

(see Section 3.2 below), and thus

$$\delta Q[P] \;=\; \mathrm{Tr}\big(\big[a,\Lambda_\mathrm{o}[a^\dagger,P]\big]-\big[a^\dagger,\Lambda_\mathrm{o}[a,P]\big]\big) \;+\; 3\,\mathrm{Tr}\,\big[P\,,\,\Lambda_\mathrm{o}[a,P][a^\dagger,P]-\Lambda_\mathrm{o}[a^\dagger,P][a,P]\big]\;. \tag{2.31}$$

Hence, if Λ_o is bounded, the second term vanishes and $\delta Q[P]$ reduces to the boundary term.[8]

Formally, another invariant is the rank of the projector,

$$r \;:=\; \mathrm{rank}(P) \;=\; \dim(\mathrm{im}P)\;, \tag{2.32}$$

which may differ from Q when being infinite. Let us consider the class of projectors which can be decomposed as

$$P \;=\; U\,\big(\widehat{P}\,+\,P'\big)\,U^\dagger \qquad\text{with}\qquad \widehat{P}=\bar{P}\otimes\mathbb{1}_{\mathcal{H}} \qquad\text{and}\qquad \widehat{P}\,P'=0=P'\,\widehat{P}\;, \tag{2.33}$$

where U is an admissible unitary transformation, $\bar{P}$ denotes a constant projector, and the rank r' of P' is finite. In this case the difference of r and Q is determined by the invariant $\mathrm{U}(n)$ trace [23]

$$R[P] \;=\; \mathrm{tr}\,\bar{P}\;\in\{0,1,\ldots,n\}\;, \tag{2.34}$$

and such projectors are characterized by the pair (R,Q). Formally, the total rank then becomes $r\,=\,R{\cdot}\infty+Q$. Since a and $a^\dagger$ commute with $\widehat{P}$, the topological charge depends only on the finite-rank piece,

$$Q[P] \;=\; Q[\widehat{P}{+}P'] \;=\; Q[P'] \;=\; \mathrm{Tr}\,P' \;=\; r'\;. \tag{2.35}$$

For $\bar{P}{=}0$ one obtains the abelian solutions because they fit into $\mathrm{U}(\mathcal{H})$. For $R{>}0$ we have nonabelian solutions because more than one copy of $\mathcal{H}$ is needed to accomodate them.

Any projector in $\mathbb{C}^n\otimes\mathcal{H}=\mathcal{H}^{\oplus n}$ can be parametrized as

$$P \;=\; |T\rangle\,\langle T|T\rangle^{-1}\langle T| \tag{2.36}$$

where

$$|T\rangle \;=\; \Big(|T^1\rangle \quad |T^2\rangle \quad \ldots \quad |T^r\rangle\Big) \;=\; \begin{pmatrix} |T^1_1\rangle & |T^2_1\rangle & \ldots & |T^r_1\rangle \\ |T^1_2\rangle & |T^2_2\rangle & \ldots & |T^r_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |T^1_n\rangle & |T^2_n\rangle & \ldots & |T^r_n\rangle \end{pmatrix} \;=\; \begin{pmatrix} |T_1\rangle \\ |T_2\rangle \\ \vdots \\ |T_n\rangle \end{pmatrix} \tag{2.37}$$

[8] Yet again, this condition is too strong a demand as the examples of (2.13) and (2.16) indicate.

denotes an $n{\times}r$ array of kets in $\mathcal{H}$, with r possibly being infinite. Thus,

$$\langle T|T\rangle \;=\; \Big(\langle T^\ell|T^m\rangle\Big) \;=\; \Big(\sum_{i=1}^n\langle T_i^\ell|T_i^m\rangle\Big) \tag{2.38}$$

stands for an invertible $r{\times}r$ matrix, and r is the rank of P. The column vectors $|T^\ell\rangle$ span the image im$P$ of the projector in $\mathcal{H}^{\oplus n}$. There is some ambiguity in the definition (2.36) of $|T\rangle$ since

$$|T\rangle \;\to\; |T\rangle\,\Gamma \qquad \text{for} \quad \Gamma\in\mathrm{GL}(r) \tag{2.39}$$

amounts to a change of basis in im$P$ and does not change the projector P. This freedom may be used to normalize $\langle T|T\rangle = \mathbb{1}_r$, which is still compatible with $\Gamma\in\mathrm{U}(r)$. On the other hand, a unitary transformation

$$|T\rangle \;\to\; U\,|T\rangle \qquad \text{for} \quad U\in\mathrm{U}(\mathcal{H}^{\oplus n}) \tag{2.40}$$

yields a unitarily equivalent projector $U P U^\dagger$. Altogether, we have the bijection

$$\tilde{P} \;=\; U\,P\,U^\dagger \qquad \Longleftrightarrow \qquad |\tilde{T}\rangle \;=\; U\,S\,|T\rangle \;=\; U\,|T\rangle\,\Gamma \qquad \text{with} \quad S\in\mathrm{GL}(\mathrm{im}P)\ . \tag{2.41}$$

Here, the trivial action of $U\in\mathrm{U}(\mathrm{im}P)\times\mathrm{U}(\ker P)$ on P can be subsumed in the S action on $|T\rangle$.

Any diagonal projector can be cast into the form

$$P_{\mathrm{d}} \;=\; \mathrm{diag}\Big(\underbrace{\mathbb{1}_{\mathcal{H}},\dots,\mathbb{1}_{\mathcal{H}}}_{R\text{ times}},\, P_Q\,,\underbrace{\mathbf{0}_{\mathcal{H}},\dots,\mathbf{0}_{\mathcal{H}}}_{n-R-1\text{ times}}\Big) \qquad \text{with} \qquad P_Q \;=\; \sum_{m=0}^{Q-1}|m\rangle\langle m|\ . \tag{2.42}$$

Formally, P_{d} has rank r in $\mathcal{H}^{\oplus n}$. A corresponding $r{\times}n$ array of kets via (2.36) would be

$$|T_{\mathrm{d}}\rangle \;=\; \begin{pmatrix} |\mathcal{H}\rangle & \emptyset & \dots & \emptyset & 0_Q \\ \emptyset & |\mathcal{H}\rangle & & \emptyset & 0_Q \\ \vdots & & \ddots & & \vdots \\ \emptyset & \emptyset & & |\mathcal{H}\rangle & 0_Q \\ \emptyset & \emptyset & \dots & \emptyset & |T_Q\rangle \\ \vdots & \vdots & & \vdots & \\ \emptyset & \emptyset & \dots & \emptyset & 0_Q \end{pmatrix} \qquad \text{with} \qquad \left\{ \begin{array}{rcl} |\mathcal{H}\rangle &=& \big(|0\rangle\ |1\rangle\ |2\rangle\ |3\rangle\ \dots\big) \\ \emptyset &=& \big(\,0\quad 0\quad 0\quad 0\quad \dots\big) \\ |T_Q\rangle &=& \big(|0\rangle\ |1\rangle\ \dots\ |Q{-}1\rangle\big) \\ 0_Q &=& \big(\,\underbrace{0\quad 0\quad \dots\quad 0}_{Q\text{ times}}\,\big) \end{array}\right. \ . \tag{2.43}$$

In the abelian case, $n{=}1$ and $R{=}0$, this reduces to $|T_{\mathrm{d}}\rangle = |T_Q\rangle$, and any projector is unitarily equivalent to P_Q. Due to (2.22), analogous results are valid in the complementary case $P\to\mathbb{1}{-}P$.

2.4 BPS solutions

Let us focus on classical solutions within a Grassmannian. For $\Phi=\mathbb{1}{-}2P$, the equation of motion (2.10) reduces to

$$0 \;=\; [\Delta P\,,\,P] \;=\; [a^\dagger,\,(\mathbb{1}{-}P)aP]+[a\,,\,Pa^\dagger(\mathbb{1}{-}P)] \;=\; [a\,,\,(\mathbb{1}{-}P)a^\dagger P]+[a^\dagger,\,Pa(\mathbb{1}{-}P)]\ . \tag{2.44}$$

Still, we do not know how to characterize its full solution space. However, (2.44) is identically satisfied by projectors subject to [24, 9]

$$\text{either the BPS equation} \qquad 0 \;=\; [a\,,P]\,P \;=\; (\mathbb{1}{-}P)\,a\,P \tag{2.45}$$

$$\text{or the anti-BPS equation} \qquad 0 \;=\; [a^\dagger,P]\,P \;=\; (\mathbb{1}{-}P)\,a^\dagger P \tag{2.46}$$

which are only "first-order". Solutions to (2.45) are called solitons while those to (2.46) are named anti-solitons. Hermitian conjugation shows that the latter are obtained from the former by exchanging $P \leftrightarrow \mathbb{1}-P$, and so we can ignore the anti-BPS solutions for most of the paper. Equation (2.45) means that

$$a \quad \text{maps} \quad \text{im}P \;\hookrightarrow\; \text{im}P \tag{2.47}$$

and, hence, characterizes subspaces of $\mathcal{H}^{\oplus n}$ which are stable under the action of a. The term "BPS equation" derives from the observation that (2.45) inserted in (2.23) implies the saturation of the BPS bound (2.24), which simplifies to

$$E[P] \;=\; 8\pi\, Q[P] \;=\; 8\pi\, \text{Tr}\big(P\, a\, (\mathbb{1}-P)\, a^\dagger P\big) \;=\; 8\pi\, \text{Tr}\big(P\, a\, a^\dagger P - a\, P\, a^\dagger\big)\ . \tag{2.48}$$

For BPS solutions of the particular form (2.33) we indeed recover that $E[P] = 8\pi\, \text{Tr}P'$. Clearly, these BPS solutions constitute the absolute minima of the energy functional within each Grassmannian.

When the parametrization (2.36) is used, the BPS condition (2.45) simplifies to

$$a\,|T_i^\ell\rangle \;=\; |T_i^{\ell'}\rangle\, \gamma_{\ell'}{}^{\ell} \qquad \text{for some } r{\times}r \text{ matrix} \quad \gamma \;=\; \big(\gamma_{\ell'}{}^{\ell}\big) \tag{2.49}$$

which represents the action of a in the basis chosen for im$P$. For instance, the ket $|T_Q\rangle$ in (2.43) indeed obeys

$$a\,|T_Q\rangle \;=\; |T_Q\rangle\, \gamma_Q \qquad \text{with} \qquad \gamma_Q \;=\; \begin{pmatrix} 0 & \sqrt{1} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2} & & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \dots & 0 & \sqrt{Q-1} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad , \tag{2.50}$$

and the diagonal projector P_{d} in (2.42) is BPS, but it is by far not the only one.

Any basis change in im$P$ induces a similarity transformation $\gamma \mapsto \Gamma\gamma\Gamma^{-1}$, which leaves P unaltered and thus has no effect on the value of the energy. Therefore, it suffices to consider γ to be of Jordan normal form. More generally, a unitary transformation (2.41) is compatible with the BPS condition (2.49) only if

$$U^\dagger a\, U\,|T\rangle \;=\; |T\rangle\, \gamma_U \qquad \text{for some } r{\times}r \text{ matrix } \gamma_U\ , \tag{2.51}$$

which implies that

$$\omega\,|T\rangle \;=\; |T\rangle\, \gamma_\omega \qquad \text{with} \qquad \gamma_\omega = \gamma_U - \gamma\ . \tag{2.52}$$

The trivially compatible transformations are those in U(im$P$)$\times$U(ker$P$), which can be subsumed in $S \in$ GL(im$P$) and lead to $\gamma_U{=}\Gamma^{-1}\gamma\Gamma$. Another obvious choice are rigid symmetries of the energy functional, as given in (2.13), (2.16), and (2.11) for $W{=}V^\dagger$. The challenging task then is to identify the nontrivial BPS-compatible unitary transformations, since these relate different a-stable subspaces of fixed dimension in $\mathcal{H}^{\oplus n}$ and thus generate the multi-soliton moduli space. For $|T_Q\rangle$ in (2.43) we shall accomplish this infinitesimally in Section 3.3.

2.4.1 Abelian solitons

Let us take a closer look at the BPS solutions of the noncommutative U(1) sigma model. All finite-energy configurations are based on $R{=}0$ and have rank $r = Q < \infty$, thus $E = 8\pi r$. The task is to solve the "eigenvalue equation"

$$a\,|T\rangle \;=\; |T\rangle\, \gamma \qquad \text{for} \qquad \gamma \;=\; \bigoplus_{s=1}^{q} \begin{pmatrix} \alpha_s & 1 & 0 & \dots & 0 \\ 0 & \alpha_s & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & & \alpha_s & 1 \\ 0 & 0 & \dots & 0 & \alpha_s \end{pmatrix} \qquad \text{with} \quad \alpha_s \in \mathbb{C}\ , \tag{2.53}$$

where the Jordan cells have sizes r_s for $s = 1, \ldots, q$, with $\sum_{s=1}^{q} r_s = r$. For a given rank r, the above matrices γ parametrize the r-soliton moduli space. The general solution (unique up to cell-wise normalization and basis changes $|T^{(s)}\rangle \to |T^{(s)}\rangle\Gamma^{(s)}$) reads

$$|T\rangle \;=\; \Bigl(|T^{(1)}\rangle \;\ldots\; |T^{(q)}\rangle\Bigr) \qquad \text{with} \qquad |T^{(s)}\rangle \;=\; \Bigl(|\alpha_s\rangle \;\; a^\dagger|\alpha_s\rangle \;\;\ldots\;\; \tfrac{1}{(r_s-1)!}(a^\dagger)^{r_s-1}|\alpha_s\rangle\Bigr) \tag{2.54}$$

and is based on the coherent states (2.14). In the star-product picture, the corresponding Φ represents r lumps centered at positions α_s with degeneracies r_s in the xy plane. Lifting a degeneracy by "point-splitting", the related Jordan cell dissolves into different eigenvalues. Hence, the generic situation has $r_s{=}1$ $\forall s$, and so

$$\gamma \;=\; \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r) \qquad \Longleftrightarrow \qquad |T\rangle \;=\; \Bigl(|\alpha_1\rangle \; |\alpha_2\rangle \;\ldots\; |\alpha_r\rangle\Bigr)\;, \tag{2.55}$$

which yields the projector

$$P \;=\; \sum_{k,\ell=1}^{r} |\alpha_k\rangle \Bigl(\langle\alpha_\cdot|\alpha_\cdot\rangle\Bigr)^{-1}_{k\ell} \langle\alpha_\ell| \;. \tag{2.56}$$

Each solution can be translated via a unitary transformation mediated by $D(\beta)$, which shifts $\alpha_\ell \mapsto \alpha_\ell{+}\beta$ $\forall\ell$, and rotated by $R(\vartheta)$, which moves $\alpha_\ell \mapsto \mathrm{e}^{\mathrm{i}\vartheta}\alpha_\ell$ $\forall\ell$. The individual values of α_ℓ (the soliton locations) may also be moved around by appropriately chosen unitary transformations, so that any r-soliton configuration can be reached from the diagonal one, which describes r solitons on top of each other at the coordinate origin [9]:

$$|T\rangle \;=\; U\,|T_r\rangle\,\Gamma \qquad \Longleftrightarrow \qquad P \;=\; U\,P_r\,U^\dagger \qquad \text{with} \quad P_r \;=\; \sum_{m=0}^{r-1} |m\rangle\langle m| \;. \tag{2.57}$$

We illustrate the latter point with the example of $r = 2$. Generically,

$$\bigl(|\alpha_1\rangle\;|\alpha_2\rangle\bigr) \;=\; U\,S\,\bigl(|0\rangle\;|1\rangle\bigr) \;=\; U\,\bigl(|0\rangle\;|1\rangle\bigr)\,\Gamma \qquad \text{with} \quad U\,S \;=\; |\alpha_1\rangle\langle 0| + |\alpha_2\rangle\langle 1| + \ldots\;, \tag{2.58}$$

where the omitted terms annihilate $|0\rangle$ and $|1\rangle$. Factorizing US yields

$$\begin{aligned} U \;&=\; \frac{1}{\sqrt{1{-}|\sigma|^2}}\,\bigl(|\alpha_1\rangle\;|\alpha_2\rangle\bigr)\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\gamma} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\gamma'}\end{pmatrix}\begin{pmatrix}-\sin\beta' & \cos\beta'\\ \sin\beta & -\cos\beta\end{pmatrix}\begin{pmatrix}\langle 0|\\ \langle 1|\end{pmatrix} \;+\;\ldots \\ \text{and} \qquad \Gamma \;&=\; \begin{pmatrix}\cos\beta & \cos\beta'\\ \sin\beta & \sin\beta'\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\gamma} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\gamma'}\end{pmatrix}\;, \\ \text{with} \qquad \sigma \;&=\; \langle\alpha_1|\alpha_2\rangle \;=\; \mathrm{e}^{\bar\alpha_1\alpha_2-\frac12|\alpha_1|^2-\frac12|\alpha_2|^2} \;=\; \mathrm{e}^{-\mathrm{i}(\gamma-\gamma')}\cos(\beta{-}\beta')\;, \end{aligned} \tag{2.59}$$

so that $|\sigma|^2 = \mathrm{e}^{-|\alpha_1-\alpha_2|^2}$, and $\beta{+}\beta'$ and $\gamma{+}\gamma'$ remain undetermined. The rank-2 projector becomes

$$P \;=\; \frac{1}{1{-}|\sigma|^2}\Bigl(|\alpha_1\rangle\langle\alpha_1| + |\alpha_2\rangle\langle\alpha_2| - \sigma|\alpha_1\rangle\langle\alpha_2| - \bar\sigma|\alpha_2\rangle\langle\alpha_1|\Bigr) \;=\; U\,\Bigl(|0\rangle\langle 0| + |1\rangle\langle 1|\Bigr)\,U^\dagger\;. \tag{2.60}$$

2.4.2 Nonabelian solitons

We proceed to infinite-rank projectors. For simplicity, let us discuss the case of U(2) solitons – the results will easily generalize to U(n). Clearly, we can imbed two finite-rank BPS solutions (with R=0) into U($\mathcal{H}{\oplus}\mathcal{H}$) by letting each act on a different copy of $\mathcal{H}$. Such configurations are

noncommutative deformations of the trivial projector $\bar P = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ and thus represent a combination of abelian solitons. Therefore, we turn to projectors with $R = 1$ and so, formally, $r = \infty + Q$. Given (2.33) such solutions may be considered as noncommutative deformations of the U(2) projector $\bar P = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. For this reason, one expects the generic solution $|T\rangle$ to (2.49) to be combined from a full set of states for the first copy of $\mathcal{H}$ and a finite set of coherent states $|\alpha_\ell\rangle$ in the second copy of $\mathcal{H}$,

$$|T\rangle \;=\; \begin{pmatrix} |\mathcal{H}\rangle & 0 & 0 & \dots & 0 \\ \emptyset & |\alpha_1\rangle & |\alpha_2\rangle & \dots & |\alpha_Q\rangle \end{pmatrix} \qquad \Longrightarrow \qquad P \;=\; \mathbb{1}_{\mathcal{H}} \oplus \sum_{k,\ell=1}^{Q} |\alpha_k\rangle \Big(\langle\alpha_.|\alpha_.\rangle\Big)^{-1}_{k\ell} \langle\alpha_\ell| \;, \tag{2.61}$$

with $E = 8\pi Q$. This projector is of the special form (2.33), with $\bar P = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $U = \mathbb{1}$. By a unitary transformation on the second copy of $\mathcal{H}$ such a configuration can be mapped to the diagonal form

$$|T_{\mathrm{d}}\rangle \;=\; \begin{pmatrix} |\mathcal{H}\rangle & 0 & 0 & \dots & 0 \\ \emptyset & |0\rangle & |1\rangle & \dots & |Q{-}1\rangle \end{pmatrix} \qquad \Longrightarrow \qquad P_{\mathrm{d}} \;=\; \mathbb{1}_{\mathcal{H}} \,\oplus\, P_Q \;. \tag{2.62}$$

It is convenient to reorder the basis of $\mathrm{im}P_{\mathrm{d}}$ such that

$$|T_{\mathrm{d}}\rangle \;=\; \begin{pmatrix} 0 & 0 & \dots & 0 & |\mathcal{H}\rangle \\ |0\rangle & |1\rangle & \dots & |Q{-}1\rangle & \emptyset \end{pmatrix} \;=\; \begin{pmatrix} S_Q \\ P_Q \end{pmatrix} |\mathcal{H}\rangle \;=:\; \hat T_{\mathrm{d}}\,|\mathcal{H}\rangle \;, \tag{2.63}$$

where

$$P_Q \;=\; \sum_{m=0}^{Q-1} |m\rangle\langle m| \qquad \text{and} \qquad S_Q \;=\; (a\,\tfrac{1}{\sqrt{N}})^Q \;=\; \sum_{m=Q}^{\infty} |m{-}Q\rangle\langle m| \tag{2.64}$$

denotes the Qth power of the shift operator. The form of (2.63) suggests to pass from states $|T\rangle = \begin{pmatrix} |T_1\rangle \\ |T_2\rangle \end{pmatrix}$ with $R{=}1$ to operators $\hat T = \begin{pmatrix} \hat T_1 \\ \hat T_2 \end{pmatrix}$ on $\mathcal{H}$:

$$|T\rangle \;=\; \hat T\,|\mathcal{H}\rangle \qquad \Longrightarrow \qquad P \;=\; \hat T\,(\hat T^\dagger \hat T)^{-1} \hat T^\dagger \;. \tag{2.65}$$

In fact, it is always possible to introduce $\hat T$ as

$$\hat T_i \;=\; \sum_{\ell=1}^{r} |T_i^\ell\rangle\langle \ell{-}1| \qquad \Longleftrightarrow \qquad |T_i^\ell\rangle \;=\; \hat T_i\,|\ell{-}1\rangle \qquad \text{for} \quad i=1,2 \quad \text{and} \quad \ell=1,\dots,r \;. \tag{2.66}$$

We may even put $\hat T^\dagger \hat T = \mathbb{1}_{\mathcal{H}}$ by using the freedom $\hat T \to \hat T\,\hat\Gamma$ with an operator $\hat\Gamma = (\hat T^\dagger \hat T)^{-1/2}$. Our example of $|T_{\mathrm{d}}\rangle$ in (2.63) is already normalized since

$$S_1\,S_1^\dagger \;=\; \mathbb{1}_{\mathcal{H}} \qquad \text{but} \qquad S_1^\dagger S_1 \;=\; \mathbb{1}_{\mathcal{H}} - |0\rangle\langle 0| \qquad \Longrightarrow \qquad S_Q^\dagger S_Q + P_Q \;=\; \mathbb{1}_{\mathcal{H}} \;. \tag{2.67}$$

It is instructive to turn on a BPS-compatible unitary transformation in (2.33). In our example (2.63), we apply [25]

$$U(\mu) \;=\; \begin{pmatrix} S_Q\sqrt{\frac{N_Q}{N_Q+\mu\bar\mu}}\,S_Q^\dagger & S_Q\frac{\bar\mu}{\sqrt{N_Q+\mu\bar\mu}} \\[1ex] \frac{\mu}{\sqrt{N_Q+\mu\bar\mu}}\,S_Q^\dagger & \frac{\mu\,P_Q-\sqrt{N_Q}}{\sqrt{N_Q+\mu\bar\mu}} \end{pmatrix} \qquad \text{with} \quad N_Q \;=\; {a^\dagger}^Q a^Q \;=\; N(N{-}1)\cdots(N{-}Q{+}1) \tag{2.68}$$

to $\hat{T}_{\rm d} = (S_Q, P_Q)^t$ of (2.63). With the help of

$$S_Q P_Q \;=\; 0 \;=\; N_Q P_Q \qquad \text{and} \qquad S_Q \sqrt{N_Q} \;=\; a^Q \tag{2.69}$$

we arrive at

$$\hat{T}(\mu) \;=\; U(\mu)\,\hat{T}_{\rm d} \;=\; \begin{pmatrix} a^Q \\ \mu \end{pmatrix} \frac{1}{\sqrt{N_Q+\mu\bar{\mu}}} \;=\; \begin{pmatrix} a^Q \\ \mu \end{pmatrix} \hat{\Gamma} \;=:\; \check{T}(\mu)\,\hat{\Gamma}\ . \tag{2.70}$$

This transformation can be regarded as a regularization of $\hat{T}_{\rm d}$ since

$$\lim_{\mu\to 0} \hat{T}(\mu) \;=\; \big(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\delta} \end{smallmatrix}\big)\,\hat{T}_{\rm d} \quad \text{with} \quad \delta = \lim_{\mu\to 0} \arg\mu \qquad \text{and thus} \qquad \lim_{\mu\to 0} P(\mu) \;=\; P_{\rm d}\ , \tag{2.71}$$

but note the singular normalization in the limit! For completeness we also display the transformed projector,

$$P(\mu) \;=\; U(\mu) \begin{pmatrix} \mathbb{1}_{\mathcal{H}} & \mathbf{0}_{\mathcal{H}} \\ \mathbf{0}_{\mathcal{H}} & P_Q \end{pmatrix} U^\dagger(\mu) \;=\; \begin{pmatrix} a^Q \frac{1}{N_Q+\mu\bar{\mu}} a^{\dagger Q} & a^Q \frac{\bar{\mu}}{N_Q+\mu\bar{\mu}} \\ \frac{\mu}{N_Q+\mu\bar{\mu}} a^{\dagger Q} & \frac{\mu\bar{\mu}}{N_Q+\mu\bar{\mu}} \end{pmatrix}\ . \tag{2.72}$$

How do we see that such projectors are BPS? Writing $|T\rangle = \hat{T}|\mathcal{H}\rangle$, the BPS condition

$$a\,|T_i^\ell\rangle \;=\; |T_i^{\ell'}\rangle\,\gamma_{\ell'}{}^{\ell} \qquad \text{implies} \qquad a\,\hat{T}_i \;=\; \hat{T}_i\,\hat{\gamma} \qquad \text{with} \quad i=1,2 \tag{2.73}$$

for some operator $\hat{\gamma}$ in $\mathcal{H}$. We do not know the general solution for arbitrary $\hat{\gamma}$. However, an important class of solutions arises for the choice $\hat{\gamma} = a$ where the BPS condition reduces to the "holomorphicity condition"[9]

$$[\,a\,,\,\hat{T}_i\,] \;=\; 0 \qquad \text{for} \quad i=1,2\ . \tag{2.74}$$

These equations are satisfied by *any* set of functions $\{\hat{T}_1, \hat{T}_2\}$ of a alone, i.e. not depending on $a^\dagger$. Indeed, for the example of $\hat{T}_{\rm d}$ in (2.63), we concretely have

$$\hat{\gamma} \;=\; a\,P_Q \;+\; (\mathbb{1}_{\mathcal{H}} {-} P_Q)\,a\,\sqrt{\tfrac{N-Q}{N}}\ , \tag{2.75}$$

while $\check{T}(\mu)$ in (2.70) is obviously holomorphic and thus BPS. Because $\hat{T}(\mu)$ emerges from $\hat{T}_{\rm d}$ via a BPS-compatible unitary transformation it shares the topological charge Q and the energy $E = 8\pi Q$ with the latter.

The generalization to arbitrary values of n and $R<n$ is straightforward: $\hat{T}$ becomes an $n{\times}R$ array of operators $(\hat{T}_i^L)$, on which left multiplication by a amounts to right multiplication by an $R{\times}R$ array of operators $\hat{\gamma}$. For the special choice $\hat{\gamma}_{L'}{}^{L} = \delta_{L'}{}^{L} a$, any collection of holomorphic functions of a serves as a solution for $\hat{T}_i^L$. Quite generally, one can show that for polynomial functions $\hat{T}_i^L(a)$ the resulting projector has a finite topological charge Q given by the degree of the highest polynomial and is thus of finite energy [26, 5, 27]. In this formulation it becomes evident that nonabelian solutions have a smooth commutative limit, where $\sqrt{2\theta}a \to z$ and $\theta \to 0$. Indeed, they are seen as deformations of the well known solitons in the $\mathrm{Gr}(n,R) = \frac{\mathrm{U}(n)}{\mathrm{U}(R)\times\mathrm{U}(n-R)}$ Grassmannian sigma model.[10] Hence, the moduli space of the nonabelian solitons coincides with that of their commutative cousins. By rescaling $\hat{T} \to \hat{T}\Gamma$ with a $\mathbb{C}$-valued $R{\times}R$ matrix Γ we

[9]This may even be the general case: If there exists an invertible operator $\hat{\Gamma}$ solving $a\hat{\Gamma} = \hat{\Gamma}\hat{\gamma}$, then the general solution to (2.73) reads $\hat{T}_i = \check{T}_i\hat{\Gamma}$ with $[a, \check{T}_i] = 0$, and $\hat{\Gamma}$ can be scaled to unity.

[10]For the above-discussed example one has $\mathrm{Gr}(2,1) = \mathbb{C}P^1$.

can eliminate R complex parameters from nR independent polynomials. For a charge-Q solution, there remain $nRQ+(n{-}1)R$ complex moduli, of which $(n{-}1)R$ parametrize the vacuum and nRQ describe the position and shape of the multi-soliton [26]. For the case of $\mathbb{C}P^1$ this yields a complex $2Q$ dimensional soliton moduli space represented by $\left(\begin{smallmatrix}\check{T}_1\\ \check{T}_2\end{smallmatrix}\right)=\left(\begin{smallmatrix}a^Q+\ldots+\nu\\ \lambda a^Q+\ldots+\mu\end{smallmatrix}\right)$. Our sample calculation above suggests that taking $\check{T}_2\to\mu$ and then performing the limit $\mu\to 0$ one recovers the complex Q dimensional moduli space of the abelian solitons given by $|T\rangle=\big(|\alpha_1\rangle\ \ldots\ |\alpha_{Q-1}\rangle\big)$ as a boundary.

2.5 Some non-BPS solutions

For the record we also present a particular class of non-BPS (and non-anti-BPS) Grassmannian solutions to the equation of motion $[\Delta P,P]=0$. From the action of Δ on a basis operator,

$$\Delta\,|m\rangle\langle n|\;=\;(m{+}n{+}1)\,|m\rangle\langle n|-\sqrt{mn}\,|m{-}1\rangle\langle n{-}1|-\sqrt{(m{+}1)(n{+}1)}\,|m{+}1\rangle\langle n{+}1|\;, \tag{2.76}$$

we infer that Δ maps the k-th off-diagonal into itself, so in particular it retains the diagonal:

$$\Delta\,|m\rangle\langle m|\;=\;(2m{+}1)\,|m\rangle\langle m|-m\,|m{-}1\rangle\langle m{-}1|-(m{+}1)\,|m{+}1\rangle\langle m{+}1|\;. \tag{2.77}$$

It follows that *every* diagonal projector is a solution. Let us first consider the case of U(1). Given the natural ordering of the basis $\{|m\rangle\}$ of $\mathcal{H}$, any diagonal projector P' of finite rank $r{=}Q$ can be written as

$$P'\;=\;\sum_{s=1}^{q}\sum_{k=0}^{r_s-1}|m_s{+}k\rangle\langle m_s{+}k|\;=\;\sum_{s=1}^{q}S^\dagger_{m_s}\,P_{r_s}\,S_{m_s}\qquad\text{with}\quad m_{s+1}>m_s{+}r_s\quad\forall s \tag{2.78}$$

and $\sum_{s=1}^q r_s=r$. Via

$$\begin{aligned}\Delta P'\;=\;\sum_{s=1}^{q}\Big\{&m_s\,|m_s\rangle\langle m_s|\;-\;m_s\,|m_s{-}1\rangle\langle m_s{-}1|\\ &-\;(m_s{+}r_s)\,|m_s{+}r_s\rangle\langle m_s{+}r_s|\;+\;(m_s{+}r_s)\,|m_s{+}r_s{-}1\rangle\langle m_s{+}r_s{-}1|\Big\}\end{aligned} \tag{2.79}$$

its energy is easily calculated as

$$\tfrac{1}{8\pi}\,E[P']\;=\;Q\;+\;2\sum_{s=1}^{q}m_s\;, \tag{2.80}$$

which is obviously minimized for the BPS case $q=1$ and $m_1=0$. The energy is additive as long as the two projectors to be combined are not getting “too close”. This picture generalizes to the nonabelian case by formally allowing r_s and m_s to become infinite. In particular, the energy does not change when one imbeds P' (or several copies of it) into $\mathrm{U}(\mathcal{H}^{\oplus n})$ and adds to it a constant projector as in (2.33). Hence, with the proper redefinition of the m_s, (2.80) holds for nonabelian diagonal solutions as well.

The inversion $P\to\mathbb{1}{-}P$ generates additional solutions, which for the structure in (2.33) and BPS-compatible unitaries U are represented as

$$P\;=\;U\big(\widehat{P}{-}P'\big)\,U^\dagger\qquad\text{with}\qquad\widehat{P}=(\mathbb{1}_n{-}\bar{P})\otimes\mathbb{1}_{\mathcal{H}}\qquad\text{and}\qquad(\mathbb{1}{-}\widehat{P})P'=0=P'(\mathbb{1}{-}\widehat{P})\;, \tag{2.81}$$

When $\mathbb{1}{-}\widehat{P}{+}P'$ is BPS, then P becomes anti-BPS with topological charge $Q[P]=-\mathrm{Tr}P'$, producing an anti-soliton with energy $E[P]=8\pi\mathrm{Tr}P'$. It is possible to combine solitons and anti-solitons to a non-BPS solution via

$$P\;=\;P_{\mathrm{sol}}+P_{\overline{\mathrm{sol}}}\qquad\text{provided}\qquad P_{\mathrm{sol}}P_{\overline{\mathrm{sol}}}\;=\;0\;=\;P_{\overline{\mathrm{sol}}}P_{\mathrm{sol}}\;, \tag{2.82}$$

so that their topological charges and energies simply add to

$$Q[P] \;=\; Q[P_{\text{sol}}] + Q[P_{\overline{\text{sol}}}] \qquad \text{and} \qquad E[P] \;=\; E[P_{\text{sol}}] + E[P_{\overline{\text{sol}}}] \;. \tag{2.83}$$

For the diagonal case, this is included in the solutions discussed above. Examples for U(1) and U(2) (which appeared in [28]) are (for $m>r$)

$$\begin{aligned} P \;&=\; \mathbb{1}_{\mathcal{H}} - P_m + P_r \;=\; \mathbb{1}_{\mathcal{H}} - \sum_{k=r}^{m-1} |k\rangle\langle k| \qquad &&\text{with} \quad Q = r{-}m \quad \text{and} \quad \tfrac{1}{8\pi}E = r{+}m \;, \\ P \;&=\; P_{r_1} \oplus (\mathbb{1}_{\mathcal{H}}{-}P_{r_2}) &&\text{with} \quad Q = r_1{-}r_2 \quad \text{and} \quad \tfrac{1}{8\pi}E = r_1{+}r_2 \;, \end{aligned} \tag{2.84}$$

respectively. Besides the global translations and rotations, other unitary transformations are conceivably compatible with the equation of motion $[\Delta P, P] = 0$, generating moduli spaces of non-BPS Grassmannian solutions. Outside the Grassmannian manifolds, many more classical configurations are to be found.

3 Fluctuation analysis

In order to investigate the stability of the classical configurations constructed in the previous section, we must study the energy functional E in the neighborhood of the solution under consideration. Since the latter is a minimum or a saddle point of E, all the linear stability information is provided by the Hessian, i.e. the second variation of E evaluated at the solution. The Hessian is viewed as a linear map on the solution's tangent space of fluctuations, and its spectrum encodes the invariant information: Zero modes belong to field directions of marginal stability (and extend to moduli if they remain zero to higher orders) while negative eigenvalues signal instabilities. A perturbation in such a field direction provides (part of) the initial conditions of a runaway solution in a time extension of the model. We cannot be very specific about the stability of a general classical configuration. Therefore, we shall restrict our attention to the stability of BPS solutions (as introduced above), at which the Hessian simplifies sufficiently to obtain concrete results. Since any BPS configuration is part of a moduli space which is imbedded in some Grassmannian which itself lies inside the full configuration space of the noncommutative U(n) sigma model, the total fluctuation space contains the subspace of Grassmannian fluctuations which in turn includes the subspace of BPS perturbations, the latter being zero modes associated with moduli. In order to simplify the problem of diagonalizing the Hessian we shall search for decompositions of the fluctuation space into subspaces which are invariant under the action of the Hessian. As we consider configurations of finite energy only, admissible fluctuations ϕ must render $\delta^2 E$ finite and keep the "background" Φ unitary. Furthermore, they need to be subject to the same conditions as Φ itself: ϕ is bounded, $[a, \phi]$ and $[a^\dagger, \phi]$ are Hilbert-Schmidt, and $\Delta\phi$ is traceclass. Finally, in keeping with our restricted notion of Grassmannian, we do not admit hermitian fluctuations[11] which alter the topological charge.

3.1 The Hessian

The Taylor expansion of the energy functional around some finite-energy configuration Φ reads

$$\begin{aligned} E[\Phi{+}\phi] \;&=\; E[\Phi] + \int\!\mathrm{d}^2 z \, \frac{\delta E}{\delta\Phi(z)}[\Phi]\,\phi(z) \,+\, \frac{1}{2}\int\!\mathrm{d}^2 z\!\int\!\mathrm{d}^2 z'\,\phi(z)\,\frac{\delta^2 E}{\delta\Phi(z)\,\delta\Phi(z')}[\Phi]\,\phi(z') + \ldots \\ &=:\; E[\Phi] \,+\, E^{(1)}[\Phi,\phi] \,+\, E^{(2)}[\Phi,\phi] \,+\, \ldots \;, \end{aligned} \tag{3.1}$$

[11] Perturbations inside the Grassmannian must be hermitian (see Section 3.2 below).

where the U(n) traces are included in $\int\mathrm{d}^2z$. The perturbation ϕ is to be constrained as to keep the background Φ unitary. Since we compute to second order in ϕ, it does not suffice to take $\phi\in T_\Phi\mathrm{U}(\mathcal{H}^{\oplus n})$. Rather, we must include the leading correction stemming from the exponential map onto $\mathrm{U}(\mathcal{H}^{\oplus n})$, which generates the finite perturbation $\phi=\Phi'-\Phi$. The latter is subject to the constraint

$$(\Phi^\dagger{+}\phi^\dagger)(\Phi{+}\phi) \;=\; \mathbb{1} \qquad\Longrightarrow\qquad \phi^\dagger \;=\; -\Phi^\dagger\phi\,\Phi^\dagger+\Phi^\dagger\phi\,\Phi^\dagger\phi\,\Phi^\dagger+O(\phi^3)\ , \tag{3.2}$$

which we shall use to eliminate $\phi^\dagger$ from the variations. It is important to realize that in this way the term linear in $\phi^\dagger$ generates a contribution to $E^{(2)}[\Phi,\phi]$. Performing the expansion for the concrete expression (2.8) and using (3.2) we arrive at[12]

$$\begin{aligned}
E^{(1)}[\Phi,\phi] \;&=\; \pi\,\mathrm{Tr}\bigl\{[a^\dagger,\,\Phi^\dagger\phi\,\Phi^\dagger]\,[a\,,\,\Phi]+[a^\dagger,\,\Phi]\,[a\,,\,\Phi^\dagger\phi\,\Phi^\dagger]-[a^\dagger,\,\Phi^\dagger]\,[a\,,\,\phi]-[a^\dagger,\,\phi]\,[a\,,\,\Phi^\dagger]\bigr\}\\
&=\; 2\pi\,\mathrm{Tr}\bigl\{(\Delta\Phi^\dagger\,\Phi-\Phi^\dagger\,\Delta\Phi)\,\Phi^\dagger\,\phi\bigr\} \;=\; 0\ ,
\end{aligned}\tag{3.3}$$

$$\begin{aligned}
E^{(2)}[\Phi,\phi] \;&=\; \pi\,\mathrm{Tr}\bigl\{[a^\dagger,\,\Phi^\dagger\phi\,\Phi^\dagger]\,[a\,,\,\phi]+[a^\dagger,\,\phi]\,[a\,,\,\Phi^\dagger\phi\,\Phi^\dagger]\\
&\qquad -\;[a^\dagger,\,\Phi^\dagger\phi\,\Phi^\dagger\phi\,\Phi^\dagger]\,[a\,,\,\Phi]-[a^\dagger,\,\Phi]\,[a\,,\,\Phi^\dagger\phi\,\Phi^\dagger\phi\,\Phi^\dagger]\bigr\}\\
&=\; 2\pi\,\mathrm{Tr}\bigl\{\Phi^\dagger\phi\,\Phi^\dagger\phi\,\Phi^\dagger\,\Delta\Phi-\Phi^\dagger\phi\,\Phi^\dagger\,\Delta\phi\bigr\}\\
&=\; 2\pi\,\mathrm{Tr}\bigl\{\phi^\dagger\,\Delta\phi-\phi^\dagger\,(\Phi\Delta\Phi^\dagger)\,\phi\bigr\}+O(\phi^3) \;=:\; 2\pi\,\mathrm{Tr}\bigl\{\phi^\dagger\,H\,\phi\bigr\}+O(\phi^3)\ ,
\end{aligned}\tag{3.4}$$

defining the Hessian $H=\Delta-(\Phi\Delta\Phi^\dagger)$ as a self-adjoint operator. Hence, our task is essentially reduced to working out the spectrum of the Hessian. Since Δ is clearly a positive semidefinite operator, an instability can only occur in directions for which $\langle\Phi\Delta\Phi^\dagger\rangle$ is sufficiently large.

For later reference, we present the action of H in the oscillator basis,

$$\begin{aligned}
H\,\sum_{m,\ell}\phi_{m,\ell}\,|m\rangle\langle\ell| \;&=\; \sum_{m,\ell}(H\phi)_{m,\ell}\,|m\rangle\langle\ell| \qquad\qquad \text{with}\\
(H\phi)_{m,\ell} \;&=\; (m{+}\ell{+}1)\,\phi_{m,\ell}-\sqrt{(m{+}1)(\ell{+}1)}\,\phi_{m+1,\ell+1}-\sqrt{m\ell}\,\phi_{m-1,\ell-1}\\
&\quad -\;\sum_{j,k}\Phi_{m,j}\,\bigl\{(j{+}k{+}1)\,\Phi_{j,k}-\sqrt{(j{+}1)(k{+}1)}\,\Phi_{j+1,k+1}-\sqrt{jk}\,\Phi_{j-1,k-1}\bigr\}\,\phi_{k,\ell}\ ,
\end{aligned}\tag{3.5}$$

where $\Phi_{m,\ell}$ as well as $\phi_{m,\ell}$ are still $n{\times}n$ matrix-valued. At diagonal abelian backgrounds, as given in (2.78), the matrices reduce to numbers and the latter expression simplifies to

$$\begin{aligned}
(H\phi)_{m,\ell} \;&=\; (m{+}\ell{+}1-2b_m)\,\phi_{m,\ell}-\sqrt{(m{+}1)(\ell{+}1)}\,\phi_{m+1,\ell+1}-\sqrt{m\ell}\,\phi_{m-1,\ell-1} \qquad \text{with}\\
b_m \;&=\; \sum_{s=1}^{q}\bigl\{\delta_{m,m_s}m_s+\delta_{m,m_s-1}m_s+\delta_{m,m_s+r_s}(m_s{+}r_s)+\delta_{m,m_s+r_s-1}(m_s{+}r_s)\bigr\}\ ,
\end{aligned}\tag{3.6}$$

which for fixed ℓ differs from $(\Delta\phi)_{m,\ell}$ in at most $4q$ entries. For diagonal U(1) BPS backgrounds $\Phi=\Phi_r=\mathbb{1}-2P_r$ this reduces further to

$$b_m \;=\; r\,\bigl(\delta_{m,r-1}+\delta_{m,r}\bigr)\ . \tag{3.7}$$

It is important to note that these expressions do not yet yield a matrix representation of H because the constraint (3.2) on the allowable perturbations must still be taken into account. We can do this by replacing ϕ with $\phi-\Phi\phi^\dagger\Phi$ everywhere; then ϕ becomes unconstrained.

[12] The same result is obtained by considering $E[\exp\{t\phi\}\Phi]$ up to $O(t^2)$.

3.2 Decomposition into even and odd fluctuations

We specialize to Grassmannian backgrounds, $\Phi = \mathbb{1} - 2P = \Phi^\dagger$, characterized by a hermitian projector P and obeying $\Phi^2 = \mathbb{1}$. Any such projector induces an orthogonal decomposition

$$\mathbb{C}^n \otimes \mathcal{H} \;=\; P\,(\mathbb{C}^n \otimes \mathcal{H}) \oplus (\mathbb{1}{-}P)(\mathbb{C}^n \otimes \mathcal{H}) \;=:\; \mathrm{im}P \oplus \ker P\ , \tag{3.8}$$

and a fluctuation ϕ decomposes accordingly as

$$\phi \;=\; \underbrace{P\,\phi\,P \;+\; (\mathbb{1}{-}P)\,\phi\,(\mathbb{1}{-}P)}_{\phi_{\mathrm{e}}} \;+\; \underbrace{P\,\phi\,(\mathbb{1}{-}P) \;+\; (\mathbb{1}{-}P)\,\phi\,P}_{\phi_{\mathrm{o}}}\ , \tag{3.9}$$

where the subscripts refer to "even" and "odd", respectively. Since Φ acts as $-\mathbb{1}$ on $\mathrm{im}P$ but as $+\mathbb{1}$ on $\ker P$, we infer that

$$\Phi\,\phi_{\mathrm{e}} \;=\; \phi_{\mathrm{e}}\Phi \qquad\text{and}\qquad \Phi\,\phi_{\mathrm{o}} \;=\; -\phi_{\mathrm{o}}\Phi \qquad\Longrightarrow\qquad \phi_{\mathrm{e}}^\dagger \;=\; -\phi_{\mathrm{e}} \qquad\text{and}\qquad \phi_{\mathrm{o}}^\dagger \;=\; \phi_{\mathrm{o}} \tag{3.10}$$

to leading order from (3.2), i.e. even fluctuations are anti-hermitian while odd ones are hermitian. This implies that odd fluctuations keep Φ inside its Grassmannian, but even ones perturb away from it. It also follows that in

$$\mathrm{Tr}\big(\phi^\dagger\,\Delta\phi\big) \;=\; \mathrm{Tr}\big(\phi_{\mathrm{e}}^\dagger\,\Delta\phi_{\mathrm{e}}\big) + \mathrm{Tr}\big(\phi_{\mathrm{o}}^\dagger\,\Delta\phi_{\mathrm{o}}\big) + \mathrm{Tr}\big(\phi_{\mathrm{e}}^\dagger\,\Delta\phi_{\mathrm{o}}\big) + \mathrm{Tr}\big(\phi_{\mathrm{o}}^\dagger\,\Delta\phi_{\mathrm{e}}\big) \tag{3.11}$$

the last two terms cancel each other. Furthermore, the equation of motion (2.44), $[\Delta P, P] = 0$, implies that

$$(\Phi\Delta\Phi)_{\mathrm{o}} \;=\; 0 \qquad\Longrightarrow\qquad \mathrm{Tr}\big(\phi_{\mathrm{e}}^\dagger\;\Phi\Delta\Phi\;\phi_{\mathrm{o}}\big) \;=\; 0 \;=\; \mathrm{Tr}\big(\phi_{\mathrm{o}}^\dagger\;\Phi\Delta\Phi\;\phi_{\mathrm{e}}\big)\ , \tag{3.12}$$

because only an even number of odd terms in a product survives under the trace. Combining (3.11) and (3.12) we conclude that

$$E^{(2)}[\Phi,\phi_{\mathrm{e}}{+}\phi_{\mathrm{o}}] \;=\; E^{(2)}[\Phi,\phi_{\mathrm{e}}] + E^{(2)}[\Phi,\phi_{\mathrm{o}}]\ , \tag{3.13}$$

which allows us to treat these two types of fluctuations separately.

The above decomposition has another perspective. Recall that any background configuration Φ being unitary can be diagonalized by some unitary transformation,

$$\Phi \;=\; U\,\Phi_{\mathrm{d}}\,U^\dagger \;=\; U\,\mathrm{diag}\big(\{\mathrm{e}^{\mathrm{i}\lambda_i}\}\big)\,U^\dagger\ . \tag{3.14}$$

When Φ is hermitian, i.e. inside some Grassmannian, the diagonal phase factors can be just $+1$ or -1, and U is determined only up to a factor $V \in \mathrm{U}(\mathrm{im}P) \times \mathrm{U}(\ker P)$ which keeps the two eigenspaces $\mathrm{im}P$ and $\ker P$ invariant. Adding a perturbation ϕ lifts the high degeneracy of Φ, so that the diagonalization of $\Phi{+}\phi$ requires an infinitesimal "rotation" K of $\mathrm{im}P$ and $\ker P$ inside $\mathcal{H}$ as well as a "large" rediagonalization V inside the two eigenspaces. Modulo higher order terms we may write

$$\begin{aligned}\Phi + \phi \;&=\; U\,(1{+}K)\,V\,(\Phi_{\mathrm{d}} + \phi_{\mathrm{d}})\,V^\dagger(1{-}K)\,U^\dagger \;=\; \Phi \;+\; U\big\{V\,\phi_{\mathrm{d}}\,V^\dagger \;+\; [K\,,\,\Phi_{\mathrm{d}}]\big\}U^\dagger \\ &\text{with}\qquad [V,\Phi_{\mathrm{d}}]=0 \qquad\text{and}\qquad K=-K^\dagger \quad\text{infinitesimal}\ ,\end{aligned} \tag{3.15}$$

where ϕ_{d} is a purely diagonal and anti-hermitian fluctuation. Since V depends on ϕ it should not be absorbed into U. It rather generates all non-diagonal fluctuations inside $\mathrm{im}P$ and $\ker P$, allowing us to rewrite

$$V\,\phi_{\mathrm{d}}\,V^\dagger \;=\; \phi_{\mathrm{d}}' \;+\; [\Lambda_{\mathrm{e}}\,,\,\phi_{\mathrm{d}}]\ , \qquad\text{with}\qquad \Lambda_{\mathrm{e}}^\dagger = -\Lambda_{\mathrm{e}} \tag{3.16}$$

being a generator of $\mathrm{U}(\mathrm{im}P)\times\mathrm{U}(\ker P)$ and a modified diagonal perturbation ϕ'_{d}. Redenoting also $K=\epsilon\Lambda_{\mathrm{o}}$ with a real and infinitesimal ϵ and a generator Λ_{o} of the Grassmannian, the general fluctuation is parametrized as

$$\phi \;=\; U\left\{\phi'_{\mathrm{d}} \;+\; [\Lambda_{\mathrm{e}}\,,\,\phi_{\mathrm{d}}] \;+\; [\Lambda_{\mathrm{o}}\,,\,\epsilon\Phi_{\mathrm{d}}]\right\}U^\dagger \tag{3.17}$$

and decomposed (after diagonalizing the background via U) into a "radial" part ϕ'_{d} and an "angular" part $\phi_{\mathrm{a}}=[\Lambda,\text{any}]$ with Λ generating $\mathrm{U}(\mathcal{H}^{\oplus n})$ [29]. For a Grassmannian background all terms have definite hermiticity properties, and we can identify

$$\phi_{\mathrm{e}} \;=\; U\left\{\phi'_{\mathrm{d}} \;+\; [\Lambda_{\mathrm{e}}\,,\,\phi_{\mathrm{d}}]\right\}U^\dagger \qquad\text{and}\qquad \phi_{\mathrm{o}} \;=\; U\left\{[\Lambda_{\mathrm{o}}\,,\,\epsilon\Phi_{\mathrm{d}}]\right\}U^\dagger \;=\; \epsilon\,[U\Lambda_{\mathrm{o}}U^\dagger\,,\Phi]\ . \tag{3.18}$$

We have seen in (3.13) above that the even and odd fluctuations can be disentangled in $E^{(2)}$. It is not clear, however, whether the diagonal perturbations can in turn be separated from the even angular ones in the fluctuation analysis (but see below for diagonal backgrounds where $U=\mathbb{1}$).

3.3 Odd or Grassmannian perturbations

As far as stability of BPS configurations is concerned, the odd perturbations are easily dealt with by a general argument. Since a shift by ϕ_{o} keeps Φ inside its Grassmannian, wherein Φ already minimizes the energy, such a perturbation cannot lower the energy any further and we can be sure that negative modes are absent here. Therefore, solitons in the noncommutative *Grassmannian* sigma model are stable, up to possible zero modes. An obvious zero mode is generated by the translational and rotational symmetry. A glance at (2.13) and (2.16) shows that the corresponding infinitesimal operators Λ are given by

$$\Lambda_{\mathrm{trans}} \;=\; \alpha\, a^\dagger - \bar\alpha\, a \qquad\text{and}\qquad \Lambda_{\mathrm{rot}} \;=\; \mathrm{i}\vartheta\, a^\dagger a\ , \tag{3.19}$$

which indeed leads to the annihilation of $[\Lambda_{\mathrm{trans}},\Phi]$ and $[\Lambda_{\mathrm{rot}},\Phi]$ by H as we shall see.

From the discussion at the end of Section 2.4 we know that for $r{>}1$ there are additional zero modes inside the Grassmannian, because the multi-soliton moduli spaces are higher-dimensional. Parametrizing these BPS-compatible perturbation as follows,

$$\Phi+\phi_{\mathrm{o}} \;=\; U_{\mathrm{B}}\,\Phi\,U_{\mathrm{B}}^\dagger \;=\; \Phi+\epsilon\,[\Lambda_{\mathrm{B}},\Phi]+O(\epsilon^2) \qquad\text{for}\quad U_{\mathrm{B}}=\mathrm{e}^{\epsilon\Lambda_{\mathrm{B}}}\quad\text{with}\quad \Lambda_{\mathrm{B}}^\dagger=-\Lambda_{\mathrm{B}}\ , \tag{3.20}$$

the corresponding Lie-algebra element (2.26) becomes

$$\omega \;=\; U_{\mathrm{B}}^\dagger\,[a\,,U_{\mathrm{B}}] \;=\; (\mathrm{e}^{-\epsilon\,\mathrm{ad}\Lambda_{\mathrm{B}}}-1)\,a \;=\; \epsilon\,[a\,,\Lambda_{\mathrm{B}}]+O(\epsilon^2)\ , \tag{3.21}$$

and the condition (2.52) of BPS compatibility to leading order in ϵ reads

$$[a\,,\Lambda_{\mathrm{B}}]\,|T\rangle \;=\; |T\rangle\,\gamma_\omega \qquad\text{for some } r{\times}r \text{ matrix } \gamma_\omega\ . \tag{3.22}$$

Let us try to find Λ_{B} in the abelian case by perturbing around the diagonal BPS configuration

$$|T_r\rangle \;=\; \big(|0\rangle\ |1\rangle\ \ldots\ |r{-}1\rangle\big) \qquad\Longleftrightarrow\qquad \Phi_r \;=\; \mathbb{1}_{\mathcal{H}} \;-\; 2\sum_{m=0}^{r-1}|m\rangle\langle m|\ . \tag{3.23}$$

Expanding the generator

$$\Lambda_{\mathrm{B}} \;=\; \sum_{m,\ell}\Lambda_{m,\ell}\,|m\rangle\langle\ell| \qquad\text{with}\quad \bar\Lambda_{m,\ell}=-\Lambda_{\ell,m} \tag{3.24}$$

it is easy to see that the even part of $\Lambda_{\rm B}$ automatically fulfils (3.22), and so restrictions to $\Lambda_{m,\ell}$ arise only for the odd components. In fact, only the terms with $m{\ge}r$ and $\ell{<}r$ in the above sum may violate (3.22), which leads to the conditions

$$\sqrt{m{+}1}\,\Lambda_{m+1,\ell}-\sqrt{\ell}\,\Lambda_{m,\ell-1} \;=\; 0 \qquad \text{for all} \quad m\geq r \quad \text{and} \quad \ell<r\ . \tag{3.25}$$

Since $\ell{=}0$ yields $\Lambda_{m+1,0}{=}0$ as a boundary condition, this hierarchy of equations puts most components to zero, except for $m=r,\dots,r{+}\ell$ at any fixed $\ell{<}r$. These remaining $r(r{+}1)/2$ components are subject to $r(r{-}1)/2$ equations from (3.25), whose solution

$$\Lambda_{r+j,\ell+j} \;=\; \sqrt{\tfrac{(\ell+j)(\ell+j-1)\cdots(\ell+1)}{(r+j)(r+j-1)\cdots(r+1)}}\,\Lambda_{r,\ell} \qquad \text{for} \quad j=1,\dots,r{-}1{-}\ell \quad \text{at} \quad \ell\leq r{-}2 \tag{3.26}$$

fixes $r(r{-}1)/2$ components in terms of the $r{-}1$ components appearing on the right hand side, which therefore are free complex parameters. The rth free parameter $\Lambda_{r,r-1}$ does not enter and is associated with the rigid translation mode.[13] Finally, the ensueing BPS perturbation (3.20) is found to be

$$\phi_{\rm o} \;=\; \epsilon\,[\Lambda_{\rm B},\Phi_r] \;=\; \epsilon\sum_{m=r}^{\infty}\sum_{\ell=0}^{r-1}\Bigl(\Lambda_{m,\ell}\,|m\rangle\langle\ell| \;+\; \bar{\Lambda}_{m,\ell}\,|\ell\rangle\langle m|\Bigr)\ , \tag{3.27}$$

with $\Lambda_{m,\ell}$ taken from (3.26). To higher orders in ϵ, the BPS-compatibility condition is not automatically satisfied by our solution (3.26) but this can be repaired by adding suitable even components to $\Lambda_{\rm B}$. Our result ties in nicely with the observation of r complex moduli α_k in (2.56) whose shifts produce precisely r complex zero modes. For the simplest non-trivial case of $r{=}2$, one can also extract these modes from differentiating (2.59) with respect to α_1 or α_2.

3.4 Even or non-Grassmannian perturbations

The stability analysis of BPS configurations inside the full noncommutative U(n) sigma model requires the investigation of the even fluctuations $\phi_{\rm e}$ as well. Here, we have only partial results to offer.[14] Yet, there is the following general argument which produces an unstable even fluctuation mode $\phi_{\rm neg}$ (but not an eigenmode) for *any* noncommutative multi-soliton $\Phi=\mathbb{1}-2P$ with $Q>0$. For this, consider some other multi-soliton $\tilde{\Phi}=\mathbb{1}-2\tilde{P}$ which is contained in Φ in the sense that

$$\text{im}(\tilde{P}) \;\subset\; \text{im}(P) \qquad \Longleftrightarrow \qquad \tilde{P}\,P \;=\; P\,\tilde{P} \;=\; \tilde{P}\ . \tag{3.28}$$

We then simply say that $\tilde{P}\subset P$. It follows that their difference Π is the orthogonal complement of $\tilde{P}$ in im(P),

$$\Pi \;=\; P-\tilde{P} \;\subset\; P \qquad \Longrightarrow \qquad \Pi^2=\Pi \qquad \text{and} \qquad \Pi\,\tilde{P}=0=\tilde{P}\,\Pi\ . \tag{3.29}$$

In particular, we may choose $\tilde{P}=0$. For any such pair $(P,\tilde{P})$ there exists a continuous path

$$\begin{aligned} \Phi(s) \;&=\; {\rm e}^{{\rm i}s\Pi}(\mathbb{1}{-}2P) \;=\; \mathbb{1}-2P+(1{-}{\rm e}^{{\rm i}s})\Pi \;=\; \mathbb{1}-2\tilde{P}-(1{+}{\rm e}^{{\rm i}s})\Pi \\ \text{connecting} \qquad &\Phi(0)=\Phi=\mathbb{1}{-}2P \qquad \text{with} \qquad \Phi(\pi)=\tilde{\Phi}=\mathbb{1}{-}2\tilde{P}\ . \end{aligned} \tag{3.30}$$

Note that $\Phi(s)$ interpolates between two different Grassmannians, touching them only at $s{=}0$ and $s{=}\pi$. Since we assumed that Φ and $\tilde{\Phi}$ are BPS we know that

$$E[\Phi] \;=\; 8\pi\,Q \qquad \text{and} \qquad E[\tilde{\Phi}] \;=\; 8\pi\,\tilde{Q} \tag{3.31}$$

[13] The rigid rotation mode is absent because Φ_r is spherically symmetric.
[14] Even for the commutative sigma model this is an open problem [2].

with Q and $\tilde{Q}$ being the topological charges of P and $\tilde{P}$, respectively.[15] Inserting (3.30) into the expression (2.8) for the energy and abbreviating $1-\mathrm{e}^{\mathrm{i}s} = \rho$ we compute

$$\begin{aligned} E(s) \ := \ E[\Phi(s)] \ &= \ 8\pi\, Q - 2\pi(\rho+\bar{\rho})\, \mathrm{Tr}\big([a,P][\Pi,a^\dagger] + [a^\dagger,P][\Pi,a]\big) + 2\pi\, \rho\bar{\rho}\, \mathrm{Tr}\big([a,\Pi][\Pi,a^\dagger]\big) \\ &= \ 8\pi\, Q - 4\pi(1-\cos s)\, \mathrm{Tr}\big([a,P][\Pi,a^\dagger] + [a^\dagger,P][\Pi,a] - [a,\Pi][\Pi,a^\dagger]\big) \ , \end{aligned} \tag{3.32}$$

where the two traces could be combined due to the relation

$$\rho + \bar{\rho} \ = \ \rho\bar{\rho} \ = \ 2(1-\cos s) \ . \tag{3.33}$$

Luckily, we do not need to evaluate the traces above. Knowing that $E(\pi) = 8\pi\tilde{Q}$ we infer that the last trace in (3.32) must be equal to $Q-\tilde{Q}$ and hence

$$\begin{aligned} E(s) \ &= \ 8\pi\, Q \ - \ 4\pi(1-\cos s)(Q-\tilde{Q}) \ = \ 8\pi\big(Q\, \cos^2\tfrac{s}{2} \ + \ \tilde{Q}\, \sin^2\tfrac{s}{2}\big) \\ &= \ 8\pi\, Q \ - \ 2\pi(Q-\tilde{Q})\, s^2 \ + \ O(s^4) \ . \end{aligned} \tag{3.34}$$

Evidently, for $Q>0$ we can always lower the energy of a given BPS configuration by applying an even perturbation $\phi_{\text{neg}} = -\mathrm{i}\epsilon\Pi$ towards a BPS solution with smaller charge $\tilde{Q} < Q$. The higher the charge of P the more such modes are present. Yet, they are not independent of one another but rather span a cone extending from the background, as we shall see in examples below. In fact, there is no reason to expect any of these unstable fluctuations to represent an eigenmode of the Hessian, and in general they do not. Nevertheless, their occurrence again demonstrates that there must be (at least) one negative eigenvalue of H, and for diagonal U(1) BPS backgrounds we shall prove in Section 4.3 that there is exactly one. The only stable BPS solutions are therefore the "vacua" defined by $P'{=}0$ in (2.33) and based on a constant $\mathrm{U}(n)$ projector $\bar{P}$. This leaves no stable solutions in the abelian case besides $\Phi = \mathbb{1}_{\mathcal{H}}$.

In addition to the unstable mode, there exist also a number of non-Grassmannian zero modes around each BPS configuration, which generate nearby non-BPS solutions to the equation of motion. This will become explicit in the examples discussed below.

4 Perturbations of U(1) backgrounds

4.1 Invariant subspaces

We specialize further to diagonal U(1) backgrounds Φ as given by (2.78) (not necessarily BPS). Using (3.5) it is easy to see that H maps any off-diagonal into itself. Let us parametrize the k-th upper diagonal $\mathcal{D}_k$ as

$$\phi_{(+k)} \ = \ \sum_{m=0}^{\infty} \mu_{(k)m}\, |m\rangle\langle m{+}k| \qquad \text{with} \quad \mu_{(k)m} \in \mathbb{C} \qquad \text{for} \quad k = 0,1,2,\dots \ . \tag{4.1}$$

The hermiticity properties (3.10) of the perturbations demand that we combine $\mathcal{D}_k$ and $\mathcal{D}_k^\dagger$ into a subspace $\mathcal{E}_k$ of the k-th upper plus lower diagonals by defining

$$\mathcal{E}_k \ := \ \big\{\phi_{(k)} \ \big|\ \phi_{(k)} \ = \ \phi_{(+k)} - \Phi\, \phi_{(+k)}^\dagger \Phi\big\} \ , \qquad \text{which implies} \qquad \phi_{(k)}^\dagger \ = \ -\Phi\, \phi_{(k)}\, \Phi \ . \tag{4.2}$$

The direct sum of all $\mathcal{E}_k$ is the full admissible tangent space to $\mathrm{U}(\mathcal{H})$, and so we may decompose

$$\phi \ = \ \sum_{k=0}^{\infty} \phi_{(k)} \ = \ \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \Big\{ \mu_{(k)m}\, |m\rangle\langle m{+}k| \ \mp \ \bar{\mu}_{(k)m}\, |m{+}k\rangle\langle m| \Big\} \ , \tag{4.3}$$

[15] Note that $\tilde{Q}$ need not be smaller than Q when $r(P)$ is infinite.

with the sign depending on whether the component is even or odd. Since H maps $\mathcal{E}_k$ into itself, the bilinear form defined in (3.4) is block-diagonal on the set of $\mathcal{E}_k$,

$$2\pi\,\mathrm{Tr}\big\{\phi^\dagger_{(k)}\,H\,\phi_{(\ell)}\big\} \;\sim\; \delta_{k\ell}\ , \tag{4.4}$$

which implies the factorization

$$E^{(2)}\,[\Phi,\textstyle\sum_k\phi_{(k)}] \;=\; \textstyle\sum_k E^{(2)}\,[\Phi,\phi_{(k)}]\ . \tag{4.5}$$

In other words, $\mathcal{E}_k$ forms an H-invariant subspace for each value of k. In particular, $\mathcal{E}_0$ is the space of admissible skew-hermitian diagonal matrices, and the (purely imaginary) diagonal fluctuations $\phi_{(0)}\equiv\phi_{\mathrm{d}}\in\mathcal{E}_0$ can be considered on their own.

4.2 Results for diagonal U(1) backgrounds

Apparently, for diagonal U(1) backgrounds it suffices to study the second variation form $E^{(2)}[\Phi,.]$ on each subspace $\mathcal{E}_k$ ($k\geq 0$) separately. Let us give more explicit formulae for the restriction of $E^{(2)}[\Phi,.]$ to these subspaces. To this end, we denote the non-zero entries of Φ by $\delta_j:=\Phi_{jj}$ ($j=0,1,2,\dots$). We have $\delta_j=\pm1$ for each j, and the set

$$J \;:=\; \{j\geq 0\mid \delta_{j+1}\neq\delta_j\} \tag{4.6}$$

is finite. Then, a straightforward calculation shows that the restriction of $E^{(2)}[\Phi,.]$ to

$$\begin{aligned}
&\mathcal{E}_0 \;=\; \Big\{\phi_{(0)}\;\Big|\;\phi_{(0)}\;=\;\mathrm{i}\sum_{m=0}^{\infty}\phi_m\,|m\rangle\langle m|\quad\text{with}\quad\phi_m\in\mathbb{R}\Big\}\qquad\text{is given by}\\
&\tfrac{1}{2\pi}E^{(2)}[\Phi,\phi_{(0)}] \;=\; \sum_{m=0}^{\infty}(m{+}1)\,(\phi_{m+1}-\phi_m)^2-2\sum_{j\in J}(j{+}1)\,(\phi_{j+1}^2+\phi_j^2)\ .
\end{aligned}\tag{4.7}$$

The formula for $E^{(2)}[\Phi,.]$ on $\mathcal{E}_{k>0}$ is more complicated. Considering a fixed k-th upper diagonal $\mathcal{D}_k$ as parametrized in (4.1), we first evaluate (suppressing the subscript (k))

$$\mathrm{Tr}\{\phi^\dagger_{(+k)}\,\Delta\,\phi_{(+k)}\} \;=\; \sum_{m=0}^{\infty}|\sqrt{m{+}1}\,\mu_{m+1}-\sqrt{m{+}k{+}1}\,\mu_m|^2 \;=\; \tfrac12 k\,|\mu_0|^2+\tfrac12\sum_{m=0}^{\infty}R_m(\mu)\ , \tag{4.8}$$

with

$$\begin{aligned}
R_m(\mu) \;&:=\; |\sqrt{m{+}1}\,\mu_{m+1}-\sqrt{m{+}k{+}1}\,\mu_m|^2+|\sqrt{m{+}k{+}1}\,\mu_{m+1}-\sqrt{m{+}1}\,\mu_m|^2\\
&=\;\big(\sqrt{m{+}k{+}1}-\sqrt{m{+}1}\big)^2\big(|\mu_{m+1}|^2+|\mu_m|^2\big)+2\sqrt{m{+}k{+}1}\sqrt{m{+}1}\,|\mu_{m+1}-\mu_m|^2\ .
\end{aligned}\tag{4.9}$$

Armed with this expressions, we compute the second variation on $\mathcal{E}_k$. To state the outcome, it is useful to introduce the index sets

$$J-k \;:=\; \{j{-}k\in\mathbb{N}_0\,|\,j\in J\}\qquad\text{and}\qquad A\;:=\;(J\cup(J-k))\setminus(J\cap(J-k))\ , \tag{4.10}$$

i.e. A is the symmetric difference of J and $J{-}k$. For $\phi_{(k)}\in\mathcal{E}_k$ a direct calculation results in

$$\begin{aligned}
\tfrac{1}{2\pi}E^{(2)}[\Phi,\phi_{(k)}] \;=\;& k\,|\mu_0|^2+\sum_{j\in\mathbb{N}_0\setminus A}R_j(\mu)+\sum_{j\in A}(2j{+}2{+}k)\big(|\mu_j|^2+|\mu_{j+1}|^2\big)\\
&-2\sum_{j\in J}(j{+}1)\big(|\mu_j|^2+|\mu_{j+1}|^2\big)\;-\;2\sum_{j\in J-k}(j{+}k{+}1)|\mu_j|^2\;-\;2\sum_{j\in J-k+1}(j{+}k)|\mu_j|^2\ .
\end{aligned}\tag{4.11}$$

This expression simplifies when k is greater than the largest element of $J{+}1$. Then the sets $J{-}k$ and $J{-}k{+}1$ are empty, so $A = J$, and (4.11) takes the form

$$\tfrac{1}{2\pi}E^{(2)}[\Phi,\phi_{(k)}] \;=\; k\,|\mu_0|^2 \;+\sum_{j\in\mathbb{N}_0\setminus J} R_j(\mu) \;+\; k\sum_{j\in J}\bigl(|\mu_j|^2+|\mu_{j+1}|^2\bigr)\;. \tag{4.12}$$

One sees that for each $j{\in}J$ the corresponding coefficient μ_j decouples from all successive coefficients $\mu_{m>j}$. Since the elements of J signify in the string of μ_m the boundaries between even and odd fluctuations, this observation confirms the decomposition (3.13) of $E^{(2)}$ into an even and odd part also after the restriction to $\mathcal{E}_k$.[16] Furthermore, we note that $R_j(\mu)>0$ unless $\mu_j=\mu_{j+1}=0$, which makes it obvious from the expression (4.12) that the quadratic form $E^{(2)}$ is strictly positive on each $\mathcal{E}_k$ with $k>\max_{j\in J}(j{+}1)$. For the remaining $\mathcal{E}_k$ we have to work a little harder.

4.3 Results for diagonal U(1) BPS backgrounds

The idea is to pursue the *reduction of* $E^{(2)}$ *to a sum of squares* whenever possible. For diagonal BPS solutions (2.57)

$$\Phi_r \;=\; \mathbb{1}_{\mathcal{H}} \;-\; 2\sum_{m=0}^{r-1}|m\rangle\langle m| \;=\; \sum_{m=0}^{\infty}\delta_m\,|m\rangle\langle m| \qquad\text{with}\quad \delta_m \;=\; \begin{cases}-1 & \text{for } m<r\\ +1 & \text{for } m\ge r\end{cases}\;, \tag{4.13}$$

this strategy turns out to be successful at all $k\ge1$ but breaks down at $k=0$. We note that now $J=\{r{-}1\}$, which implies a distinction of cases: "very off-diagonal" perturbations have $k>r$, "slightly off-diagonal" perturbations occur for $1\le k\le r$, and diagonal perturbations mean $k=0$, to be discussed last. It will also be instructive to visualize the Hessian on the space $\mathcal{E}_k$ in matrix form,

$$E^{(2)}[\Phi_r,\phi_{(k)}] \;=\; 2\pi\sum_{m,\ell=0}^{\infty}\bar\mu_m\,H^{(k)}_{m\ell}\,\mu_\ell\;, \tag{4.14}$$

defining an infinite-dimensional matrix $H^{(k)}=\bigl(H^{(k)}_{m\ell}\bigr)$.

4.3.1 Very off-diagonal perturbations

Since J consists just of one element, at $k>r$ the matrix $H^{(k)}$ splits into two parts only, of which the odd one has finite size r. More explicitly,

$$H^{(k)} \;=\; H^{(k)}_{\mathrm{Gr}(P)}\oplus H^{(k)}_{\ker P}\;, \tag{4.15}$$

with

$$\tfrac12 H^{(k)}_{\mathrm{Gr}(P)} = \begin{pmatrix} k+1 & -\sqrt{1(k{+}1)} & & & & \\ -\sqrt{1(k{+}1)} & k+3 & -\sqrt{2(k{+}2)} & & & \\ & -\sqrt{2(k{+}2)} & k+5 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2r+k-3 & -\sqrt{(r{-}1)(r{+}k{-}1)} \\ & & & & -\sqrt{(r{-}1)(r{+}k{-}1)} & 2r+k-1-\boldsymbol{r} \end{pmatrix} \tag{4.16}$$

[16]For $|J|>1$ the even and/or odd part of $E^{(2)}$ is split further. In total, $E^{(2)}$ decomposes into at least $|J|{+}1$ blocks.

$$
\tfrac{1}{2}H^{(k)}_{\ker P} \;=\; \begin{pmatrix} 2r+k+1-\boldsymbol{r} & -\sqrt{(r{+}1)(r{+}k{+}1)} & & & \\ -\sqrt{(r{+}1)(r{+}k{+}1)} & 2r+k+3 & -\sqrt{(r{+}2)(r{+}k{+}2)} & & \\ & -\sqrt{(r{+}2)(r{+}k{+}2)} & 2r+k+5 & \ddots & \\ & & \ddots & \ddots & \end{pmatrix} , \qquad (4.17)
$$

where the boldface contributions disturb the systematics and originate from the $\Phi\Delta\Phi^\dagger$ term in the Hessian. The previous paragraph asserts strict positivity of H for $k>r$. Due to the finiteness of $H^{(k)}_{\mathrm{Gr}(P)}$, the Hessian on $\mathcal{E}_{k>r}$ features precisely r positive eigenvalues[17] in its spectrum. We shall argue below that $H^{(k)}_{\ker P}$ contributes a purely continuous spectrum $\mathbb{R}_+$ for any k. Thus, the above eigenvalues are not isolated but imbedded in the continuum.

It is instructive to look at the edge of the continuum. The non-normalizable zero mode of $H^{(k)}_{\mathrm{Gr}(P)}$ is explicitly given by

$$
\mu^{\mathrm{zero}}_{r+m} \;=\; \mu_r\sqrt{\frac{(r{+}k{+}1)(r{+}k{+}2)\cdots(r{+}k{+}m)}{(r{+}1)(r{+}2)\cdots(r{+}m)}} \;=\; \mu_r\sqrt{\frac{(r{+}m{+}1)(r{+}m{+}2)\cdots(r{+}m{+}k)}{(r{+}1)(r{+}2)\cdots(r{+}k)}} \;, \qquad (4.18)
$$

which grows like $m^{k/2}$ when $m=0,1,2,\ldots$ gets large. Being unbounded for $k{>}0$ this infinite vector does not yield an admissible perturbation of Φ_r however: δ^2E is infinite.

4.3.2 Slightly off-diagonal perturbations

For each value of k in the range $1\le k\le r$ we already established in Section 3.3 the existence of an odd complex normalizable zero mode connected with a moduli parameter. Nevertheless, as we will show now, $E^{(2)}[\Phi_r,\phi_{(k)}]$ remains positive for $1\le k\le r$ albeit not strictly so. In order to simplify (4.11) we make use of the following property for $R_j(\mu)$ as defined in (4.9):

$$
\begin{aligned}
\sum_{j=m}^{l-1} R_j(\mu) \;&=\; k\,|\mu_l|^2 \;-\; k\,|\mu_m|^2 \;+\; 2\sum_{j=m}^{l-1}|\sqrt{j{+}1}\,\mu_{j+1}-\sqrt{j{+}k{+}1}\,\mu_j|^2 \\
&=\; k\,|\mu_m|^2 \;-\; k\,|\mu_l|^2 \;+\; 2\sum_{j=m}^{l-1}|\sqrt{j{+}k{+}1}\,\mu_{j+1}-\sqrt{j{+}1}\,\mu_j|^2 \;.
\end{aligned} \qquad (4.19)
$$

Here, by definition, all integers are non-negative and all sums are taken to vanish if $m\ge l$. The two equations above are easily proved by induction over l, starting from the trivial case $l=m$. We now employ the algebraic identities (4.19) to rewrite (4.11) for $1\le k\le r$ and obtain

$$
\begin{aligned}
\tfrac{1}{2\pi}E^{(2)}[\Phi_r,\phi_{(k)}] \;&=\; k\,|\mu_r|^2 \;+\; \sum_{j=r}^{\infty} R_j(\mu) \qquad (4.20)\\
&\quad +\; 2\sum_{j=0}^{r-k-2}|\sqrt{j{+}1}\,\mu_{j+1}-\sqrt{j{+}k{+}1}\,\mu_j|^2 \;+\; 2\sum_{j=r-k}^{r-2}|\sqrt{j{+}k{+}1}\,\mu_{j+1}-\sqrt{j{+}1}\,\mu_j|^2 \;,
\end{aligned}
$$

[17] meaning that the corresponding modes are normalizable eigenvectors, i.e. Hilbert-Schmidt

which is indeed positive semi-definite. We observe that the even part of $E^{(2)}$ now consists of two disjoint pieces, containing $\{\mu_0, \ldots, \mu_{r-k-1}\}$ and $\{\mu_{j\geq r}\}$, which are separated by the odd perturbations $\{\mu_{r-k}, \ldots, \mu_{r-1}\}$. If the right-hand side of (4.20) is equal to zero, then all components $\mu_{j\geq r}$ must vanish because each R_j is a strictly positive quadratic form of μ_j and μ_{j+1}. In contrast, the remaining coefficients $\mu_{j<r}$ need not vanish at the zero set of $E^{(2)}[\Phi_r, \phi_{(k)}]$. In order to find the zero modes of the Hessian, it is convenient to visualize it again in matrix form, namely

$$H^{(k)} \;=\; H^{(k)}_{\mathrm{im}P} \oplus H^{(k)}_{\mathrm{Gr}(P)} \oplus H^{(k)}_{\mathrm{ker}P} \;, \tag{4.21}$$

where the blocks have sizes $r-k$, k, and ∞, respectively. From (4.20) we extract

$$\tfrac12 H^{(k)}_{\mathrm{im}P} \;=\; \begin{pmatrix} k+1 & -\sqrt{1(k{+}1)} & & & & \\ -\sqrt{1(k{+}1)} & k+3 & -\sqrt{2(k{+}2)} & & & \\ & -\sqrt{2(k{+}2)} & k+5 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2r-k-3 & -\sqrt{(r{-}k{-}1)(r{-}1)} \\ & & & & -\sqrt{(r{-}k{-}1)(r{-}1)} & 2r-k-1-\boldsymbol{r} \end{pmatrix} \tag{4.22}$$

$$\tfrac12 H^{(k)}_{\mathrm{Gr}(P)} \;=\; \begin{pmatrix} 2r-k+1-\boldsymbol{r} & -\sqrt{(r{-}k{+}1)(r{+}1)} & & & \\ -\sqrt{(r{-}k{+}1)(r{+}1)} & 2r-k+3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2r+k-3 & -\sqrt{(r{-}1)(r{+}k{-}1)} \\ & & & -\sqrt{(r{-}1)(r{+}k{-}1)} & 2r+k-1-\boldsymbol{r} \end{pmatrix} \tag{4.23}$$

and $H^{(k)}_{\mathrm{ker}P}$ being identical to the matrix in (4.17). The extreme cases are $H^{(1)}_{\mathrm{Gr}(P)} = (0)$ and $H^{(r-1)}_{\mathrm{im}P} = (0)$, while $H^{(r)}_{\mathrm{im}P}$ is empty.

We already know that $H^{(k)}_{\mathrm{ker}P}$ is strictly positive definite. As claimed earlier, its spectrum is $\mathbb{R}_+$ and purely continuous, with a non-normalizable zero mode given by (4.18). Being finite-dimensional, the other two blocks jointly yield again just r non-negative and non-isolated proper eigenvalues for fixed $k{>}0$. Interestingly, two of these eigenvalues are now zero and give us one even and one odd complex normalizable zero mode. With $\beta_k, \gamma_k \in \mathbb{C}$ the latter take the following form:[18]

[18]The even zero mode is of course absent for $k = r$.

$$H^{(k)}_{\mathrm{im}P}: \qquad (\mu_\ell)_{\mathrm{zero}} \;=\; \gamma_k \begin{pmatrix} \sqrt{1}\sqrt{2}\sqrt{3}\cdots\sqrt{r{-}k{-}2}\sqrt{r{-}k{-}1} \\ \sqrt{k{+}1}\sqrt{2}\sqrt{3}\cdots\sqrt{r{-}k{-}2}\sqrt{r{-}k{-}1} \\ \sqrt{k{+}1}\sqrt{k{+}2}\sqrt{3}\cdots\sqrt{r{-}k{-}2}\sqrt{r{-}k{-}1} \\ \vdots \\ \sqrt{k{+}1}\sqrt{k{+}2}\sqrt{k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{-}k{-}1} \\ \sqrt{k{+}1}\sqrt{k{+}2}\sqrt{k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{-}1} \end{pmatrix}, \tag{4.24}$$

$$H^{(k)}_{\mathrm{Gr}(P)}: \qquad (\mu_\ell)_{\mathrm{zero}} \;=\; \beta_k \begin{pmatrix} \sqrt{r{+}1}\sqrt{r{+}2}\sqrt{r{+}3}\cdots\sqrt{r{+}k{-}2}\sqrt{r{+}k{-}1} \\ \sqrt{r{-}k{+}1}\sqrt{r{+}2}\sqrt{r{+}3}\cdots\sqrt{r{+}k{-}2}\sqrt{r{+}k{-}1} \\ \sqrt{r{-}k{+}1}\sqrt{r{-}k{+}2}\sqrt{r{+}3}\cdots\sqrt{r{+}k{-}2}\sqrt{r{+}k{-}1} \\ \vdots \\ \sqrt{r{-}k{+}1}\sqrt{r{-}k{+}2}\sqrt{r{-}k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{+}k{-}1} \\ \sqrt{r{-}k{+}1}\sqrt{r{-}k{+}2}\sqrt{r{-}k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{-}1} \end{pmatrix}. \tag{4.25}$$

Altogether, there are $2r{-}2$ real zero modes in $H_{\mathrm{im}P}$ and $2r$ real zero modes in $H_{\mathrm{Gr}(P)}$. Identifying $\mu_{(k)j} = \Lambda_{j,k+j} = -\bar{\Lambda}_{k+j,j}$, the latter precisely agree with the BPS moduli found in (3.26). The former zero modes correspond to moduli outside the Grassmannian, thus generating nearby non-diagonal non-BPS solutions to the equation of motion.

4.3.3 Diagonal perturbations

We come to the diagonal (or radial) perturbations $\phi_{(0)} \in \mathcal{E}_0$ (see (4.7)). The transformation of

$$\tfrac{1}{2\pi}E^{(2)}[\Phi_r,\phi_{(0)}] \;=\; \sum_{m=0}^{\infty}(m{+}1)\,(\phi_{m+1}-\phi_m)^2 - 2r\,(\phi_r^2+\phi_{r-1}^2) \tag{4.26}$$

to a sum of squares is much easier than that of (4.11), but the result shows one minus in the signature:

$$\tfrac{1}{2\pi}E^{(2)}[\Phi_r,\phi_{(0)}] \;=\; -r\,(\phi_{r-1}+\phi_r)^2 \;+ \sum_{j\geq 0,\, j\neq r-1}(j{+}1)\,(\phi_{j+1}-\phi_j)^2\ . \tag{4.27}$$

However, we can use this expression to conclude that the second variation form $E^{(2)}$ cannot have more than one negative mode on $\mathcal{E}_0$.

Lemma. *Let $V\subset\mathcal{E}_0$ be a real vector subspace such that $E^{(2)}[\Phi,\phi]<0$ for all nonzero $\phi\in V$. Then V is at most one-dimensional. In other words, one cannot find linearly independent vectors $\phi,\psi\in\mathcal{E}_0$ such that $E^{(2)}[\Phi,.]$ takes negative values on each non-zero linear combination of ϕ and ψ.*

Proof. A necessary and sufficient condition for a pair of linearly independent vectors $\phi,\psi\in\mathcal{E}_0$ to span a negative subspace for $E^{(2)}(.) := E^{(2)}[\Phi,.]$ is that

$$E^{(2)}(\phi)<0\ ,\quad E^{(2)}(\psi)<0 \quad\text{and}\quad B(\phi,\psi)^2-E^{(2)}(\phi)E^{(2)}(\psi)<0\ , \tag{4.28}$$

where $B(\phi,\psi) := \frac12\{E^{(2)}(\phi{+}\psi)-E^{(2)}(\phi)-E^{(2)}(\psi)\}$ is the corresponding bilinear form. If $\phi,\psi\in\mathcal{E}_0$ satisfy these inequalities, then the same holds for their truncations (that is, vectors obtained by replacing all the coordinates ϕ_j, ψ_j with $j\geq n$ by zero) if n is large enough. Thus, we may assume

the existence of n such that $\phi_j = \psi_j = 0$ for $j \geq n$. But the restriction of $E^{(2)}$ to the space $\mathbb{R}^n$ of all vectors $\phi = (\phi_0, \phi_1, \dots, \phi_{n-1}, 0, 0, \dots)$ has signature $(1, n-1)$ according to (4.27) and, therefore, this space contains no two-dimensional negative subspaces. $\square$

For an alternative visualization, we write the second-order fluctuation form as

$$E^{(2)}[\Phi_r, \phi_{(0)}] \;=\; 2\pi \sum_{m,\ell=0}^{\infty} \phi_m \, H^{(0)}_{m\ell} \, \phi_\ell \;, \tag{4.29}$$

where the matrix entries $H^{(0)}_{m\ell}$ can be extracted from (3.6) or from (4.27) as

$$\tfrac{1}{2\pi} H^{(0)}_{m\ell} \;=\; (2m{+}1 - 2r\delta_{m,r-1} - 2r\delta_{m,r})\, \delta_{m,\ell} - m\, \delta_{m,\ell+1} - (m{+}1)\, \delta_{m+1,\ell} \;. \tag{4.30}$$

In matrix form this reads as $(m, \ell \in \mathbb{N}_0)$

$$\begin{aligned} (H^{(0)}_{m\ell}) \;&=\; \begin{pmatrix} 1 & -1 & & & \\ -1 & 3 & -2 & & \\ & -2 & 5 & -3 & \\ & & -3 & 7 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \;-\; 2\,r\,(\delta_{m,r-1}\delta_{\ell,r-1} \;+\; \delta_{m,r}\delta_{\ell,r}) \\ &=\; \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 2r{-}3 & -r{+}1 & & & \\ & -r{+}1 & \mathbf{-1} & -r & & \\ & & -r & \mathbf{+1} & -r{-}1 & \\ & & & -r{-}1 & 2r{+}3 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \;. \end{aligned} \tag{4.31}$$

In distinction to the earlier cases, this infinite Jacobian (i.e. tri-diagonal) matrix does not split. We shall compute its spectrum in a moment. Prior to this, some observations can already be made.

First of all, the unique zero mode of $H^{(0)}$ is found from (4.18) for $k{=}0$ and given by $(\gamma_0 \in \mathbb{R})$

$$\phi_{\text{zero}} \;=\; \mathrm{i}\gamma_0\, \Phi_r \qquad \Longleftrightarrow \qquad (\phi_m)_{\text{zero}} \;=\; \gamma_0\, (\underbrace{-1, \dots, -1}_{r \text{ times}}, +1, +1, \dots) \;, \tag{4.32}$$

which clearly generates the global phase rotation symmetry $\Phi \to \mathrm{e}^{\mathrm{i}\gamma_0}\Phi$ already noted in (2.11). In contrast to the zero modes in $\mathcal{E}_{k>0}$ depicted in (4.18), (4.24) and (4.25), ϕ_{zero} is not Hilbert-Schmidt but bounded. Although it does not belong to a proper zero eigenvalue of the Hessian, ϕ_{zero}, being proportional to Φ_r, still meets all our requirements for an admissible perturbation.

In order to identify unstable modes we turn on all diagonal perturbations in a particular manner suggested by (3.30),

$$\Phi(\{\alpha_m\}) \;=\; \mathbb{1} \;-\; \sum_{m=0}^{r-1} (\mathrm{e}^{\mathrm{i}\alpha_m}{+}1)\, |m\rangle\langle m| \;+\; \sum_{m=r}^{\infty} (\mathrm{e}^{\mathrm{i}\alpha_m}{-}1)\, |m\rangle\langle m| \;. \tag{4.33}$$

The energy of this configuration is easily calculated to be

$$\tfrac{1}{8\pi}\, E(\{\alpha_m\}) \;=\; \sin^2 \tfrac{\alpha_0-\alpha_1}{2} + \dots + (r{-}1) \sin^2 \tfrac{\alpha_{r-2}-\alpha_{r-1}}{2} + r \cos^2 \tfrac{\alpha_{r-1}-\alpha_r}{2} + (r{+}1) \sin^2 \tfrac{\alpha_r-\alpha_{r+1}}{2} + \dots \tag{4.34}$$

with a single $\cos^2$ appearing only in the indicated place. Expanding up to second order in the real parameters α_m and putting $\alpha_m = 0$ for $m \geq r$, we find that

$$H^{(0)} < 0 \qquad \Longleftrightarrow \qquad (\alpha_0-\alpha_1)^2 + 2(\alpha_1-\alpha_2)^2 + \cdots + (r-1)(\alpha_{r-2}-\alpha_{r-1})^2 - r\,\alpha_{r-1}^2 \;\leq\; 0\ , \tag{4.35}$$

which determines a convex cone in the r-dimensional restricted fluctuation space. The most strongly negative mode is given by ($\alpha \in \mathbb{R}$)

$$\phi_{\text{neg}} \;=\; -\mathrm{i}\alpha P_r \qquad \Longleftrightarrow \qquad (\phi_m)_{\text{neg}} \;=\; -\alpha\,(\underbrace{1,\ldots,1}_{r \text{ times}},0,0,\ldots)\ . \tag{4.36}$$

Comparison with (3.34) reveals that following this mode one arrives at $\Phi = \mathbb{1}$. More generally,

$$(\phi_m) \;=\; -\alpha\,(\underbrace{0,\ldots,0}_{\tilde{r} \text{ times}},\ \underbrace{1,\ldots,1}_{(r-\tilde{r}) \text{ times}},0,0,\ldots) \tag{4.37}$$

perturbs in the direction of $\Phi = \mathbb{1} - 2P_{\tilde{r}}$. None of these modes are eigenvectors of the matrix $H^{(0)}$.

It is instructive to confirm these assertions numerically. To this end, we truncate the matrix $(H^{(0)}_{m\ell})$ at some cut-off value $m_{\max} = \ell_{\max}$ and diagonalize the resulting finite-dimensional matrix. Truncating at values $m_{\max} < r$ only allows for positive eigenvalues, whereas for values $m_{\max} \geq r$ we obtain exactly *one* negative eigenvalue. Its numerical value depends roughly linearly on r and converges very quickly with increasing cut-off as can be seen from the following graphs.

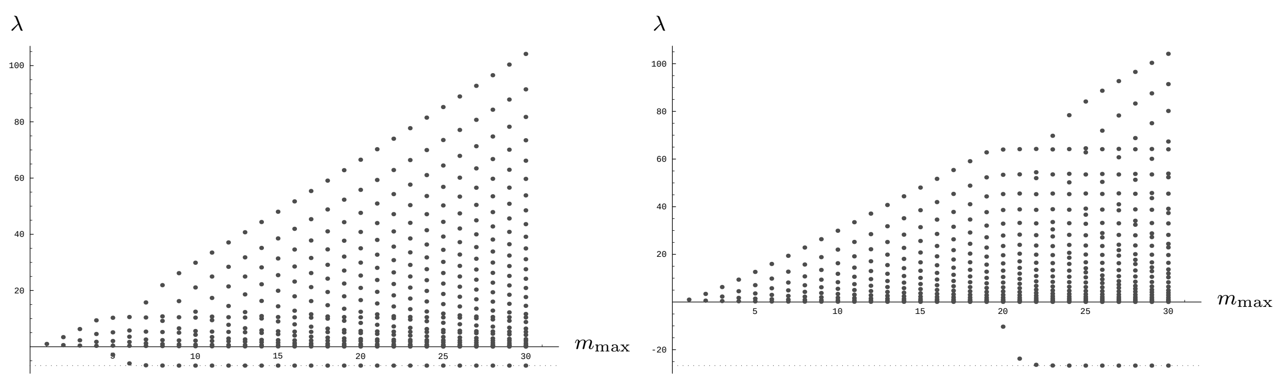


Figure 1: Possible eigenvalues λ versus the cut-off parameter $m_{\max}$ for $r = 5$ and $r = 20$

4.3.4 Spectrum of the Hessian

The spectrum of the restriction $H^{(k)} = \Delta_k - (\Phi_r \Delta \Phi_r)_k$ of the Hessian H to $\mathcal{E}_k$ can actually be calculated by passing to a basis of Laguerre polynomials. The following considerations are based on spectral theory and require all vectors to be in l_2, i.e. all operators to be Hilbert-Schmidt.

Let us first consider $\mathcal{E}_0$. We use the following property of the noncommutative Laplacian $-\Delta_0$: sending each basis vector $|m\rangle$ of $\mathcal{H}$ to the m-th Laguerre polynomial $L_m(x)$, we get a unitary map $U : \mathcal{H} \to L^2(\mathbb{R}_+, \mathrm{e}^{-x}\mathrm{d}x)$ such that $U\Delta_0 U^{-1}$ is just the operator of multiplication by x:

$$x\,L_m(x) \;=\; -m\,L_{m-1}(x) + (2m{+}1)L_m(x) - (m{+}1)L_{m+1}(x)\ , \tag{4.38}$$

which is to be compared with (4.31). Hence, in the basis of the Laguerre polynomials, the eigenvalue equation $H^{(0)}|\phi\rangle = \lambda|\phi\rangle$ is rewritten in terms of $f := U|\phi\rangle$ as

$$x\, f(x) - 2r\big(\langle f, L_{r-1}\rangle L_{r-1}(x) + \langle f, L_r\rangle L_r(x)\big) \;=\; \lambda\, f(x)\ . \tag{4.39}$$

Clearly, a function $f \in L^2(\mathbb{R}_+, \mathrm{e}^{-x}\mathrm{d}x)$ satisfies (4.39) if and only if it is given by

$$f(x) \;=\; \frac{c_0 L_{r-1}(x) + c_1 L_r(x)}{x-\lambda} \tag{4.40}$$

and the constant coefficients $c_0, c_1 \in \mathbb{R}$ satisfy the linear system

$$\begin{cases} \big(I_{r-1,r-1}(\lambda) - \frac{1}{2r}\big)\, c_0 + I_{r-1,r}(\lambda)\, c_1 \;=\; 0 \\ I_{r,r-1}(\lambda)\, c_0 + \big(I_{r,r}(\lambda) - \frac{1}{2r}\big)\, c_1 \;=\; 0 \end{cases}\ , \tag{4.41}$$

which is obtained simply by inserting (4.40) into (4.39). Here we use the notation

$$I_{k,l}(\lambda) \;:=\; \int_0^\infty \frac{\mathrm{e}^{-x}\,\mathrm{d}x}{x-\lambda}\, L_k(x)\, L_l(x) \qquad \text{for} \quad k,l \geq 0\ . \tag{4.42}$$

The integrals (4.42) are simple (though not elementary) special functions of λ. Indeed, $I_{00}(\lambda)$ is a version of the integral logarithm: $I_{00}(\lambda) = -\mathrm{e}^{-\lambda}\,\mathrm{li}(\mathrm{e}^{\lambda})$ for all $\lambda < 0$. On the other hand, using the recursion relations

$$(k{+}1)\, L_{k+1}(x) \;=\; (2k{+}1{-}x)\, L_k(x) \;-\; k\, L_{k-1}(x)\ , \tag{4.43}$$

one can show by induction over k and l that all functions $I_{k,l}(\lambda)$ are expressed in terms of $I_{00}(\lambda)$:

$$I_{kl}(\lambda) \;=\; A_{kl}(\lambda)\, I_{00}(\lambda) \;+\; B_{kl}(\lambda)\ , \tag{4.44}$$

where A_{kl} and B_{kl} are polynomials in λ of degree at most $k{+}l$. Hence, the determinant

$$F_r(\lambda) \;:=\; \begin{Vmatrix} I_{r-1,r-1}(\lambda) - \frac{1}{2r} & I_{r-1,r}(\lambda) \\ I_{r,r-1}(\lambda) & I_{r,r}(\lambda) - \frac{1}{2r} \end{Vmatrix} \tag{4.45}$$

of the linear system (4.41) is a known special function of λ, whose zeros λ_r on the negative semi-axis are precisely the negative eigenvalues of the Hessian operator $H^{(0)}[\Phi_r]$ for the diagonal BPS background of rank r.

One can now prove the existence of negative eigenvalues. By verifying that the real numbers $F_r(-\infty) := \lim_{\lambda\to-\infty} F_r(\lambda)$ and $F_r(0) := \lim_{\lambda\to 0-} F_r(\lambda)$ have different signs, we can conclude that $F_r(\lambda)$ possesses at least one zero on the negative semiaxis. For example, take $r = 1$. In this case we find

$$\left.\begin{aligned} I_{01}(\lambda) \;=\; I_{10}(\lambda) \;&=\; (1{-}\lambda) I_{00}(\lambda) - 1 \\ I_{11}(\lambda) \;&=\; (1{-}\lambda)^2 I_{00}(\lambda) + \lambda - 1 \end{aligned}\right\} \quad\Longrightarrow\quad 2\, F_1(\lambda) \;=\; -\lambda - \tfrac{1}{2} - \lambda^2 I_{00}(\lambda)\ . \tag{4.46}$$

Passing to the limit under the integral sign, we see that $F_1(-\infty) = \frac{1}{4}$ and $F_1(0) = -\frac{1}{4}$. This proves the existence of a negative eigenvalue of $H^{(0)}[\Phi_1]$.

Can $H^{(0)}[\Phi_r]$ also have non-negative eigenvalues? To disprove this possibility, we consider the eigenvalue equation (4.39) and show that it admits only the trivial solution $f(x) \equiv 0$ if $\lambda \geq 0$. Indeed, for non-negative λ the solution (4.40) is not square-integrable unless

$$c_0 L_{r-1}(\lambda) + c_1 L_r(\lambda) \;=\; 0 \qquad \Longrightarrow \qquad c_0 \;=\; c\, L_r(\lambda) \qquad \text{and} \qquad c_1 \;=\; -c\, L_{r-1}(\lambda) \tag{4.47}$$

for some constant c. Therefore, the solution (4.40) is completely determined as

$$f(x) \;=\; c\, \frac{L_r(\lambda) L_{r-1}(x) - L_{r-1}(\lambda) L_r(x)}{x-\lambda} \;=\; c \sum_{k=0}^{r-1} L_k(\lambda)\, L_k(x) \tag{4.48}$$

via the Christoffel-Darboux formula. It follows that f is orthogonal to L_r. Inserting $\langle f, L_r\rangle = 0$ into (4.39), we learn that the polynomial $(x-\lambda)f(x)$ is proportional to $L_{r-1}(x)$. But then the first equality in (4.48) demands that $L_{r-1}(\lambda) = 0$, constraining λ. Feeding this into the second expression for f in (4.48) we see that the sum runs to $r-2$ only, implying that f is orthogonal to L_{r-1} as well. This finally simplifies the eigenvalue equation (4.39) to the one for Δ_0, namely

$$x\, f(x) \;=\; \lambda\, f(x) \;, \qquad \text{whence} \qquad f(x) \;\equiv\; 0 \;, \tag{4.49}$$

as required. Hence, the non-negative part of the spectrum of $H^{(0)}$ is purely continuous.

A similar analysis can be applied to the Hessian on $\mathcal{E}_{k>0}$. In this case, the appropriate polynomials are the normalized (generalized) Laguerre polynomials L^k_m, which form an orthonormal basis for $L^2(\mathbb{R}_+, x^k \mathrm{e}^{-x} \mathrm{d}x)$. One rediscovers the r proper eigenvalues as zeroes of the characteristic polynomial for $H^{(k)}_{\mathrm{im}P} \oplus H^{(k)}_{\mathrm{Gr}(P)}$, accompanied by a continuous spectrum covering the positive semiaxis for $H^{(k)}_{\ker P}$, as claimed earlier.

If we do not care for resolving non-isolated eigenvalues, we may also infer the spectrum of the Hessian H from the classical Weyl theorem (see [30], Theorem IV.5.35 and [31], § XIII.4, Example 3).

Assertion. *Let $\Phi = \Phi_r$ be the diagonal* BPS *solution of rank $r > 0$. Then*

$$\text{spectrum}\,(H[\Phi_r]) \;=\; \{\lambda_r\} \cup [0, +\infty) \;, \tag{4.50}$$

where λ_r is a negative eigenvalue of multiplicity 1 *and* $[0, +\infty)$ *is the essential spectrum.*[19] *Furthermore, we have $-2r < \lambda_r < 0$.*

Proof. Weyl's theorem asserts that the essential spectrum is preserved under compact perturbations. The essential spectrum of Δ (on all subspaces $\mathcal{E}_k$) is known to be equal to $[0, +\infty)$. (This follows, for example, from the explicit diagonalization of Δ_k in terms of the Laguerre polynomials, as indicated above.) Since the perturbation $(\Phi_r \Delta \Phi_r)$ is finite-dimensional, Weyl's theorem applies to show that the essential spectrum of H is also equal to $[0, +\infty)$. Although we have explicitly shown that $H^{(k)}$ is positive semi-definite for $k>0$, the essential spectrum $\mathbb{R}_+$ cannot exhaust the whole spectrum because the operator $H^{(0)}$ is not positive semi-definite on $\mathcal{E}_0$. (Indeed, (4.27) shows that the quadratic form $E^{(2)}$ is negative on many finite vectors.) Hence, $H^{(0)}$ must have negative eigenvalues of finite multiplicity. The above lemma implies that there can be only one such eigenvalue, and its multiplicity must be equal to 1. This proves (4.50) and the inequality $\lambda_r < 0$. The estimate $-2r < \lambda_r$ follows because the norm of the operator $(\Phi_r \Delta \Phi_r)$ is equal to $2r$, so $H[\Phi_r] + 2r\mathbb{1}_{\mathcal{H}} \geq \Delta$ is non-negative definite and hence cannot have negative eigenvalues. $\square$

[19] We recall that the essential spectrum of a self-adjoint operator is the whole spectrum minus all isolated eigenvalues of finite multiplicity. The multiplicity of an eigenvalue is the dimension of the corresponding eigenspace.

4.4 Results for non-diagonal U(1) BPS backgrounds

Even though any non-diagonal BPS background Φ can be reached from a diagonal one by a unitary transformation U which does not change the value of the energy, the fluctuation problem will get modified under such a transformation because

$$U\,a\,U^\dagger \;=\; f(a,a^\dagger) \qquad \text{and} \qquad U\,a^\dagger U^\dagger \;=\; f^\dagger(a,a^\dagger) \tag{4.51}$$

will in general not have an action as simple as a and $a^\dagger$ in the original oscillator basis. The exceptions are the symmetry transformations, of which only the rigid translations $D(\alpha)$ and rotations $R(\vartheta)$ keep Φ within the Grassmannian. To be specific, we recall that

$$\begin{aligned} D(\alpha)^\dagger\,a\,D(\alpha) \;&=\; a+\alpha \qquad &\Longrightarrow \qquad E\left[D(\alpha)\,\Phi\,D(\alpha)^\dagger\right] \;&=\; E\left[\Phi\right]\;, \\ R(\vartheta)^\dagger\,a\,R(\vartheta) \;&=\; \mathrm{e}^{\mathrm{i}\vartheta}a \qquad &\Longrightarrow \qquad E\left[R(\vartheta)\,\Phi\,R(\vartheta)^\dagger\right] \;&=\; E\left[\Phi\right]\;. \end{aligned} \tag{4.52}$$

The invariance of the spectrum of H under translations (or rotations) can also be seen directly:

$$\begin{aligned} H\,\phi_n \;&=\; \epsilon_n\,\phi_n \qquad \text{and} \qquad \Delta\left(D\,f\,D^\dagger\right) \;=\; D\left(\Delta f\right)D^\dagger \quad \Longrightarrow \\ H'\phi'_n \;&\equiv\; \left[\Delta-(D\Phi D^\dagger)\Delta(D\Phi D^\dagger)\right]D\phi_n D^\dagger \;=\; D\left([\Delta-\Phi\Delta\Phi]\phi_n\right)D^\dagger \;=\; \epsilon_n\,\phi'_n\;. \end{aligned} \tag{4.53}$$

More general unitary transformations will not simply commute with the action of Δ and thus change the spectrum of H.

In the rank-one BPS situation, any solution is a translation of Φ_1, and hence our previous discussion of fluctuations around diagonal U(1) BPS backgrounds completely covers that case. This is no longer true for higher-rank BPS backgrounds. For a complete stability analysis of abelian r-solitons it is therefore necessary to investigate separately the spectrum of H around each soliton configuration (2.56) based on r coherent states, whose center of mass may be chosen to be the origin. Since the background holomorphically depends on r parameters $\alpha_1,\ldots,\alpha_r$ and passes to Φ_r when all parameters go to zero, one may hope to show that the spectrum of H does not change qualitatively when Φ varies inside the rank-r moduli space.

5 Perturbations of U(2) backgrounds

In this section we investigate the behavior of a simple U(2) BPS solution under perturbations and conclude that the fluctuation analysis reduces to the U(1) case discussed in the previous section.

5.1 Results for diagonal U(2) BPS backgrounds

We consider a nonabelian BPS projector of the diagonal type considered in (2.62):

$$P \;=\; \mathbb{1}_{\mathcal{H}} \oplus P_Q \qquad \text{where} \qquad P_Q \;=\; \sum_{k=0}^{Q-1} |k\rangle\langle k|\;. \tag{5.1}$$

Clearly this describes a BPS solution

$$\Phi \;=\; \begin{pmatrix} \Phi^{(1,1)} & \Phi^{(1,2)} \\ \Phi^{(2,1)} & \Phi^{(2,2)} \end{pmatrix} \;=\; \begin{pmatrix} -\mathbb{1}_{\mathcal{H}} & 0 \\ 0 & \mathbb{1}_{\mathcal{H}}-2P_Q \end{pmatrix} \tag{5.2}$$

of energy $E = 8\pi Q$. Inserting this into the expression (3.4) for the quadratic energy correction $E^{(2)}$ one obtains

$$\begin{aligned} E^{(2)}[\Phi,\phi] \;&=\; 2\pi\,\mathrm{Tr}\bigl\{\phi^\dagger\,\Delta\phi \;-\; \phi^\dagger\,(\Phi\Delta\Phi^\dagger)\,\phi\bigr\} \\ &=\; 2\pi\,\mathrm{Tr}\bigl\{\phi^\dagger\,\bigl(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\bigr)\,\Delta\phi \;-\; \phi^\dagger\,\bigl(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\bigr)\,2Q\bigl(|Q{-}1\rangle\langle Q{-}1| + |Q\rangle\langle Q|\bigr)\,\phi\bigr\}\ . \end{aligned} \tag{5.3}$$

Since the action of the Hessian is obviously diagonal, we find that

$$\phi \;=\; \begin{pmatrix}\phi^{(1,1)} & \phi^{(1,2)}\\ \phi^{(2,1)} & \phi^{(2,2)}\end{pmatrix} \qquad\Longrightarrow\qquad E^{(2)}[\Phi,\phi] \;=\; \sum_{i,j=1}^{2} E^{(2)}\bigl[\Phi^{(i,i)},\phi^{(i,j)}\bigr]\ , \tag{5.4}$$

which reduces the fluctuation analysis to a collection of abelian cases. For the case at hand, the Hessian in the $(1,.)$ sectors is given by the Laplacian and thus non-negative, while in each $(2,.)$ sector it is identical to the Hessian for the abelian rank-Q diagonal BPS case. Hence, the relevant results of the previous section carry over completely.

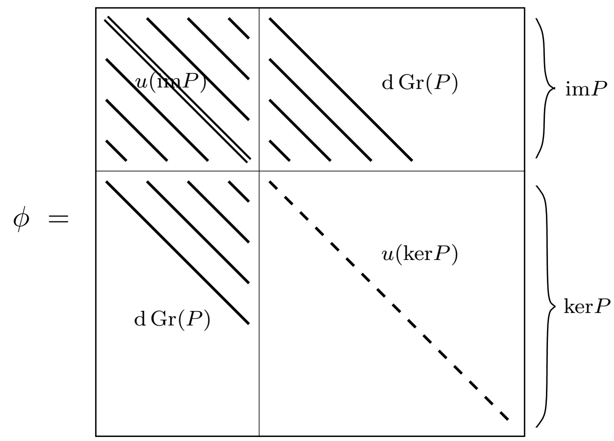
This mechanism extends to any diagonal U(2) background (not necessarily BPS), mapping the U(2) fluctuation spectrum to a collection of fluctuation spectra around corresponding diagonal U(1) configurations. The fluctuation problem around non-diagonal backgrounds, in contrast, is not easily reduced to an abelian one, except when the background is related to a diagonal one by a rigid symmetry, as defined in (2.11), (2.13) and (2.16). Only a small part of the moduli, however, is generated by rigid symmetries, as our example of $U(\mu)$ in (2.68) demonstrates even for $Q{=}1$. Finally, it is straightforward to extend these considerations to the general U(n) case.

6 Conclusions

After developing a unified description of abelian and nonabelian multi-solitons in noncommutative Euclidean two-dimensional sigma models with U(n) or Grassmannian target space, we have analyzed the issue of their stability. Thanks to the BPS bound, multi-solitons in Grassmannian sigma models are always stable. As in the commutative case, their imbedding into a unitary sigma model renders them unstable however, as there always exists one negative eigenvalue of the Hessian which triggers a decay to the vacuum configuration.

Our results are concrete and complete for abelian and nonabelian Q-soliton configurations which are diagonal in the oscillator basis (or related to such by global symmetry). For this case, we proved that the spectrum of the Hessian consists of the essential spectrum $[0,\infty)$ and an eigenvalue λ_Q of multiplicity one with $-2Q < \lambda_Q < 0$. This assertion was confirmed numerically, and the value of λ_Q was given as a zero of a particular function composed of monomials in λ and the special function $\mathrm{e}^{-\lambda}\,\mathrm{li}(\mathrm{e}^{\lambda})$.

Furthermore, the complete set of zero modes of the Hessian was identified. Each abelian diagonal Q-soliton background is characterized by a diagonal projector P of rank Q, whose image and kernel trigger a decomposition of the space of fluctuations into three invariant subspaces, namely $u(\mathrm{im}P)$, $u(\mathrm{ker}P)$ and $\mathrm{d\,Gr}(P)$. In addition, every side diagonal together with its transpose is seperately invariant under the action of the Hessian. This leads to a particular distribution of the admissible zero modes of the Hessian, displayed here for the example of $Q{=}4$:



where the double line denotes the single negative eigenvector, each solid diagonal segment represents a real normalizable zero mode, the dashed line depicts an admissible non-normalizable zero mode, and empty areas do not contain admissible zero modes. In addition, each side diagonal features a non-admissible zero mode at the edge of the continuous part $[0,\infty)$ of the spectrum. We plot the complete spectrum of the Hessian at Q=4 (cut off at size $m_{\max}$=30) for each invariant subspace $\mathcal{E}_k^{\mathrm{Gr}(P)}$ (boxes), $\mathcal{E}_k^{\mathrm{im}P}$ (stars), $\mathcal{E}_k^{\ker P}$ (crosses) and $\mathcal{E}_0$ (circles), up to k=6:

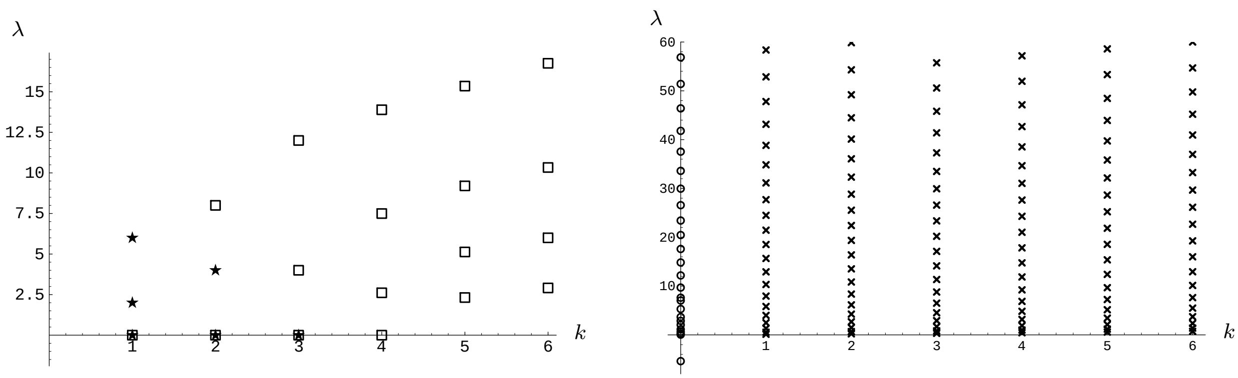


Figure 2: Eigenvalues λ of the cut-off Hessian (size 30) for Q=4 in subspaces $\mathcal{E}_k$

Since most soliton moduli are not associated with global symmetries, our explicit results for diagonal backgrounds do not obviously extend to generic (non-diagonal) backgrounds. We have not been able to compute the fluctuation spectrum across the entire soliton moduli space. The only exception is the abelian single-soliton solution which always is a translation of the diagonal configuration and therefore fully covered by our analysis. Already for the case of two U(1) solitons, the unitary transformation which changes their distance in the noncommutative plane is only partially known.

This leads us to a number of unsolved problems. The most pressing one seems to be the extension of our fluctuation analysis to non-diagonal backgrounds. Next, following the even zero modes one can now find new non-BPS solutions. Also, it is worthwhile to investigate the commutative limit of the Hessian and its spectrum. Another interesting aspect is the existence of infinite-rank abelian

projectors, i.e. via an infinite array of coherent states, associated with BPS solutions of infinite energy. Furthermore, some technical questions concerning the admissible set of fields and their fluctuations have remained. Finally, it would be rewarding to extend the geometrical understanding of sigma models to the noncommutative realm.

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